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Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

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ASYMPTOTIC SOLUTION OF A DISPERSIVE HYPERBOLIC EQUATION WITH VARIABLE COEFFICIENTS

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Prepared for
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BEDFORD, MASSACHUSETTS

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Abstract

In this paper we consider initial-boundary value problems for an energy conserving dispersive hyperbolic equation, the Klein-Gordon equation. This equation contains the main feature of dispersion: The speed of propagation depends on the frequency.

The primary purpose of this paper is to compare the asymptotic expansion of solutions obtained by a technique which we call the ray method with the asymptotic expansion of the exact solution. In every case considered, the solutions agree.

Solutions are obtained for a series of initial-boundary value problems in one space dimension with variable coefficients. As a by-product of our work we find a new feature which we call space-time diffraction. This phenomenon has the following physical interpretation: A portion of the energy of a wave reaches a boundary surface and then gradually leaks off, leaving a diminishing residue on the boundary for all time.

Introduction

In the past few years there has been a great deal of interest in the asymptotic expansion of solutions of boundary value problems and initial-boundary value problems for partial differential equations. These expansions are obtained for large distance and/or time or for large values of a parameter which may appear "naturally" in the equation and/or initial conditions, boundary conditions, etc.

Many of these problems can be treated by a method in which special curves, called "rays", in space or space-time plan an essential role. The asymptotic solution is determined in terms of "amplitude" and "phase" functions which satisfy ordinary differential equations on these rays. These equations can often be solved explicitly, so that the solution to the original problem is known if the "initial values" on the rays of the amplitude and phase functions are known. In some cases these initial values can be determined directly from the data (i.e., initial conditions, boundary conditions, etc.) of the given problem, while in other situations an indirect method is necessary. This method involves the exact solution of a "canonical problem", i.e., a problem simpler than that given originally but having the same local features as the given problem.

This "ray method" has been studied extensively by J.B. Keller and his associates at New York University for the reduced wave equation and related elliptic differential equations. A significant advantage of the method is that it eliminates the necessity of first finding the exact solution in order to find

the asymptotic expansion. Consequently, many problems for which the exact solution is not known, can be treated by the ray method. In particular, it has led to a "geometrical theory of diffraction" [1].

Recently, the ray method has been extended to time dependent problems and, in particular, to dispersive hyperbolic equations [2]. One example of this class is the Klein-Gordon equation which may be used to describe the propagation of electromagnetic waves in certain plasmas [2]. Initial-boundary value problems for this equation are presently being studied, since it is the simplest example of a large class of dispersive hyperbolic equations which conserve energy. It clearly exhibits the main feature of dispersion: The speed of propagation varies with frequency.

As of now there is no general proof that the ray method yields the asymptotic expansion of the exact solution. Consequently, a major goal of the present research is to "check" the solutions obtained by this method by comparing them with the asymptotic expansion of the exact solution in special cases where the latter can be obtained. Such checks have previously been obtained for the reduced wave equation [1,3,4]. Here they are undertaken for initial-boundary value problems for the Klein-Gordon Equation. In every case examined the asymptotic solution obtained by the ray method agrees exactly with the expansion of the exact solution.

We shall consider the Klein-Gordon equation in one space dimension with non-constant coefficients. Initial-boundary value problems for the homogeneous equation are solved by the ray method and the asymptotic expansion of the Green's function is also obtained. The distinguishing feature in the initial-boundary value problems considered, is the nature of the dependence of the initial data on the large parameter.

A method for obtaining the asymptotic expansion of the exact solution is also developed and applied to each of the problems for which a solution was obtained by the ray method. As previously noted, the solutions agree in each case.

As a by-product of our work we find a new feature which we call space-time diffraction. This phenomenon occurs in a region of space-time in which the leading term of the asymptotic expansion is zero. The name, space-time diffraction, is suggested by the similarity to an analogous problem for the reduced wave equation [3] but it differs in that it is a time-dependent phenomenon, with the following physical interpretation: A portion of the energy of a wave reaches a boundary surface and then gradually leaks off, leaving a diminishing residue on the boundary for all time.

(1) Introduction to the Ray Method

The partial differential equation to be studied is

$$(1) \qquad u_{xx} - u_{tt} - \lambda^2 b^2(x) u = 0.$$

We seek the first term of the asymptotic expansion of $u(t,x)$ for large λ.

(1.1) The Dispersion Equation and the Transport Equation

The solution will be assumed to be of the form

$$(2) \quad u(t,x) = \exp\left[i\lambda s(t,x)\right] \left[z(t,x)\right] \left[1 + O(\tfrac{1}{\lambda})\right].$$

When (2) is substituted into (1) and the coefficients of λ^2 and λ respectively are set equal to zero, the resulting equations are

$$(3) \qquad s_x^2 - s_t^2 + b^2(x) = 0$$

and

$$(4) \quad 2(s_x z_x - s_t z_t) + z \Box s = 0, \quad \Box = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}$$

We define

$$(5) \qquad k = s_x, \; \omega = -s_t$$

and then (3) becomes

$$(6) \qquad k^2 - \omega^2 + b^2(x) = 0.$$

If $b(x)$ were constant and the plane wave solution to (1) were sought, one would obtain (6) immediately. This algebraic equation is called the dispersion relation. The first order partial differential equation, (3) (which is equivalent to (6)), will be called the dispersion equation. (4) will be called the transport equation.

(1.2) The Solution of the Dispersion Equation

The method of characteristics [5] will be employed to solve the dispersion equation. The characteristic equations are

(7) $\dot{x} = 2k,\ \dot{t} = 2\omega,\ -\dot{\omega} = 0,\ \dot{k} = -2b(x)b'(x).$

The solution of this system of ordinary differential equations defines curves in (t,x) space, which will be called rays. The latter pair of equations in (7) are used to determine ω and k on the rays. The equation for ω yields

(8) $$\omega = \text{constant}.$$

Equation (6) then gives $k = k(x,\omega)$. With x as independent variable on the rays, the time, t, and the phase, s, can be shown to satisfy the following ordinary differential equations by using (5) and (7):

(9) $$\frac{dt}{dx} = \frac{\omega}{k(x,\omega)}\ ;$$

(10) $$\frac{ds}{dx} = s_x + s_t\frac{dt}{dx} = k - \omega\frac{dt}{dx}\ .^*$$

---

* If (9) and (6) were used in (10) we would have
$$\frac{ds}{dx} = k(x,\omega)-\omega\frac{\omega}{k(x,\omega)} = \frac{k^2(x,\omega)-\omega^2}{k(x,\omega)} = -\frac{b^2(x)}{k(x,\omega)}\ .$$
This result has the virtue of expressing $\frac{ds}{dx}$ completely in terms of the independent variable on the ray. However, for comparison later in the paper, the form of equation (10) best serves our purpose.

The expression $\frac{k(x,\omega)}{\omega}$, obtained by inverting (9), will be referred to as the group speed.

(1.2.1) Ray Construction and Phase for a One Dimensional Initial Manifold.

If s were given on some curve in (t,x) space, one could in general find $s(t,x)$ for all (t,x). As a special case, let us suppose that $s(0,x) = s_o(x)$ is given. This initial data for s could be described parametrically by

$$(11) \qquad t = 0,\ x = \sigma;\quad s = s_o(\sigma).$$

The method used to obtain $s_o(\sigma)$ from a given problem for $u(t,x)$ will be explained later in the discussion. Here we shall only show how $s(t,x)$ is obtained once $s_o(\sigma)$ is known. From (5) and (11)

$$(12) \qquad k(\sigma,\omega) = s_o'(\sigma).$$

Then (6), (8) and (12) yield

$$(13) \quad \omega = h(s_o'(\sigma),\ \sigma) = \pm\, h_o(s_o'(\sigma),\ \sigma);\quad h_o(k,\sigma) = \sqrt{k^2 + b^2(\sigma)}\ .^{*}$$

* We note at this point, the following convention. When a specific choice of sign is dictated, $+h_o$ or $-h_o$ will be written, but ω^2 will be written instead of h_o^2 .

With ω given by (13) k can be determined from (6) up to a choice of sign. By requiring (12) to be satisfied at $x = \sigma$ this ambiguity is eliminated yielding the solution

(14) $$k(x,\omega) = [\operatorname{sgn}(s_o'(\sigma)]\sqrt{\omega^2 - b^2(x)}.$$

The solution to (9) for which $t > 0$ is given by

(15) $$t = \int_{x_<}^{x_>} \frac{h_o}{\sqrt{\omega^2 - b^2(\eta)}}\, d\eta; \quad x_< = \min(x,\sigma),\ x_> = \max(x,\sigma).$$

With this result and (14) the solution to (10) becomes

(16) $$s(t,x) = s_o(\sigma) + \operatorname{sgn}\omega \left\{ \int_{x_<}^{x_>} \sqrt{\omega^2 - b^2(\eta)}\, d\eta - h_o t \right\}.$$

To obtain (15) and (16) we have also used the fact that $\operatorname{sgn}(x-\sigma)\operatorname{sgn} k = \operatorname{sgn}\omega$ which is a consequence of the requirement that $t > 0$. If $s^{\pm}$ are the solutions associated with $\operatorname{sgn}\omega = \pm 1$ then (16) becomes

(17) $$s^{\pm}(t,x) = s_o(\sigma) \pm \left\{ \int_{x_<}^{x_>} \sqrt{\omega^2 - b^2(\eta)}\, d\eta - h_o t \right\}.$$

Equation (15) yields a one parameter (σ) family of rays and (17) gives s in terms of the parameter. (Note that $\omega^2 = h_o^2$ is given by (13).)

<u>(1.2.2) Ray Construction and Phase for a Zero Dimensional Initial Manifold</u>

Another problem for which the solution, $s(t,x)$, will be required is the problem where s is given only at a point $(0,x_o)$. The solutions of interest will be those for which the rays emanate from the point $(0,x_o)$. We assume the given data to be $s(0,x_o) = s_o$.

From the dispersion relation it follows that

$$(18) \qquad k(x_o,\omega) = \pm\sqrt{\omega^2-b^2(x_o)}\ ; \quad k(x,\omega) = \pm\sqrt{\omega^2-b^2(x)}\ .$$

The signs of $k(x_o,\omega)$ and $k(x,\omega)$ are always chosen so that they agree for any particular ray. If σ is replaced by x_o and $s_o(\sigma)$ is replaced by s_o in (15) and (17), the equations thus obtained will be the equations of the rays and phase for the zero dimensional initial manifold.* In this case ω is a free parameter, $\omega^2 \geq b^2(x_o)$. As ω varies, the initial value of the group speed varies from -1 to 1. Note that the comment following (17) is no longer valid – σ is not the parameter on the rays – and the initial value of k is now given by (15).

(1.3) Solution of the Transport Equation

In [2] it is shown that the transport equation can be written as an ordinary differential equation along the rays. To apply the results of that paper to the discussion below it is assumed that (15), the equation for the rays, is invertible and gives x as a function of t and the parameter. It is also shown in [2] that z at time t is related to z at time t_1 by

$$(19) \qquad z(t) = z(t_1)\left[\frac{j(t_1)}{j(t)}\right]^{\frac{1}{2}} .$$

* These equations, with the indicated substitutions, are given by (1.30) and (1.31) below.

If the rays are viewed as equations defining a transformation of points of a parameter space to points $x(t)$ for fixed t, then $j(t)$ is the jacobian of that transformation:

$$(20) \qquad j(t) = \left|\frac{\partial x}{\partial \sigma}\right| \quad \text{or} \quad j(t) = \left|\frac{\partial x}{\partial \omega}\right| .$$

The quotient $\dfrac{j(t_1)}{j(t)}$ is the jacobian of transformation by rays of points x at time t_1 to points x at time t.

(1.3.1) The One Dimensional Initial Manifold

It is assumed that for $t = 0$, $x = \sigma$; $z(0,\sigma) = z_0(\sigma)$. Again, the explanation of how the initial value is obtained will be postponed. In (15), σ is taken as parameter and $\omega = \pm h_0(\sigma)$. The equation is differentiated implicitly with respect to σ with t fixed, and one obtains a linear equation in $\frac{\partial x}{\partial \sigma}$. We note that there are three terms in this equation since the upper and lower limit of integration and the integrand are functions of σ. This equation yields

$$(21) \quad j(t(x)) = \left|\frac{\partial x}{\partial \sigma}\right| = \left|\frac{k(x,\omega)}{s_0'(\sigma)} + k(x,\omega)\left(\frac{s_0'(\sigma)s_0''(\sigma)+b(\sigma)b'(\sigma)}{\omega^2}\right)\int_\sigma^x \frac{b^2(\eta)}{k^3(\eta,\omega)}\, d\eta\right| .$$

The jacobian, $j(t(x))$, as given by (21), might be zero for some value of x. There might even be a curve in (t,x) space on which $j(t(x)) = 0$. Such curves or points are called caustics. Their effect on the solutions is discussed in [2]. For simplicity it is assumed that there are no caustics. If there were

caustics, our results would only be valid for times prior to the arrival time at the caustic.

For $(t,x) = (0,\sigma)$, $j = 1$ and $k = s_o'(\sigma)$. In (19) we take $t_1 = 0$, $t(x)$ given by (15) and $j(t(x))$ given by (21). This yields

$$(22) \qquad z(t(x)) = z_o(\sigma)\left[j(t(x))\right]^{-\frac{1}{2}}.$$

(1.3.2) The Zero Dimensional Initial Manifold

From section (1.2.2), ω must now be the parameter. In this case, the rays emanate from a point $(0,x_o)$, so that the jacobian of transformation must be zero there. If $z(t(x))$ were finite at $t = 0$, setting $t_1 = 0$ in (19) shows that $z(t(x))$ would be identically zero for $t > 0$. In order that $z(t(x))$ be finite and non-zero for $t > 0$, it must have a singularity at $t = 0$ of one half the order of the zero of $j(t)$. This order of vanishing can be determined directly from the ray equations. We assume that j can be expanded in a Taylor series with constant term zero. Then if $\frac{\partial x}{\partial \omega} > 0$ for $1 >> t > 0$ we may find the leading term by noting that

$$(23) \qquad \frac{d}{dt}[j(t)] = \frac{d}{dt}\left(\frac{dx}{d\omega}\right) = \frac{\partial}{\partial\omega}\left(\frac{dx}{dt}\right).$$

$\frac{dx}{dt}$ is given by (9) with k given by (14). We may carry out the indicated operations in (23) and evaluate at $(0,x_o)$, obtaining

$$(24) \qquad j(t) = j_o t + O(t^2);$$

$$(25) \qquad j_o = \left|b^2(x_o)\omega^{-2}k^{-1}(x_o,\omega)\right|$$

Therefore, for the solution to be finite and non-zero, z must be of order $t^{-\frac{1}{2}}$ in the neighborhood of $t = 0$. We define

$$(26) \qquad \tilde{z} = \lim_{t\to 0} z(t)t^{\frac{1}{2}}$$

and then, using (19),

$$(27)\qquad z(t(x)) = \tilde{z}\left[\frac{j_o}{j(t(x))}\right]^{\frac{1}{2}}$$

In order to find $j(t(x))$ in this case, the procedure of section (1.3.1) is repeated, but with σ replaced by x_o and $k(x,\omega)$ defined by (18); (15) is differentiated implicitly with respect to ω with t fixed. Two terms are obtained—the upper limit and the integrand of (15) are functions of ω. The solution for $j(t(x))$ obtained by this method is

$$(28)\qquad j(t(x)) = \left|\frac{\partial x}{\partial \omega}\right| = \frac{\sqrt{\omega^2 - b^2(x)}}{h_o}\int_{x_<}^{x_>} \frac{b^2(\eta)}{(\omega^2 - b^2(\eta))^{3/2}}\, d\eta.$$

Equations (25) - (28) allow us to express z as a function of x and ω on the rays.

(1.4) An Example

As an example, the leading term of the asymptotic expansion of the Riemann function for equation (1) will be found. The initial data for this problem is

$$(29)\qquad u(0,x_o) = 0 \quad u_t(0,x_o) = \delta(x-x_o).$$

Since the initial data for $u(t,x)$ vanishes except for $x = x_o$, we assume that all our rays emanate from the point $(0,x_o)$.* Therefore, the results of sections (1.2.2) and (1.3.2) will be used.

The rays and phase respectively are given by

$$(30)\qquad t = \int_{x_<}^{x_>} \frac{h_o}{\sqrt{\omega^2 - b^2(\eta)}}\, d\eta \;;\quad x_< = \min(x,x_o),\; x_> = \max(x,x_o)$$

and

$$(31)\qquad s^{\pm}(t,x) = s_o^{\pm} \pm \left\{\int_{x_<}^{x_>} \sqrt{\omega^2 - b^2(\eta)}\, d\eta - h_o t\right\}.$$

* This assumption shall be justified when the solution obtained is compared to the asymptotic expansion of the exact solution, obtained in section (2).

These formulas are obtained from (15) and (17) by replacing σ by x_o and $s_o(\sigma)$ by $s_o^{\pm}$. The choice of sign in (31) is not determined for the zero dimensional initial manifold. Since the equation is linear, the solution can be obtained by superposition:

$$(32) \qquad u(t,x) \sim \sum_{\pm} z^{\pm}(t,x) \exp\left[i\lambda s^{\pm}(t,x)\right]$$

with $s^{\pm}$ given by (31) and $z^{\pm}$ given by (27) with $\tilde{z}$ replaced by $\tilde{z}^{\pm}$.

If b were constant, we could carry out all indicated integrations in (28), (30) and (31) to obtain the solution

$$(33) \qquad u \sim \sum_{\pm} \tilde{z}^{\pm} t^{-1/2} \exp\left[i\lambda\left\{s_o^{\pm} \pm \left(\sqrt{\omega^2-b^2}\ \ |x-x_o|-h_o t\right)\right\}\right];$$

$$t = |x-x_o| \frac{h_o}{\sqrt{\omega^2-b^2}}; \qquad |x-x_o| = \frac{\sqrt{\omega^2-b^2}}{h_o}\, t.$$

$u(t,x)$ is given parametrically with parameter ω by (33) except for the constants $\tilde{z}^{\pm}$ and $s_o^{\pm}$. For constant b the exact solution can also be obtained. If this solution is expanded asymptotically and the expansion is compared with (33), $\tilde{z}^{\pm}$ and $s_o^{\pm}$ can be determined. We now assume that these values are the same for the case of variable coefficients if the constant b is replaced by $b(x_o)$. R. M. Lewis calls this technique for obtaining $\tilde{z}^{\pm}$ and $s_o^{\pm}$ the indirect method. The asymptotic expansion of the exact solution for constant coefficients has been carried out in [2]. From equations (62) and (63), section (3) of that paper,

$$(34) \qquad s_o^{\pm} = 0; \qquad \tilde{z}^{\pm} = \left(\frac{h_o}{8\pi\lambda b^2}\right)^{1/2} e^{\pm\frac{i\pi}{4}}$$

The solution for variable b, obtained by using (27), (31), (32) and (34), is given by

$$(35)\quad u \sim \frac{\cos\left\{\lambda\left[\int_{x_<}^{x_>} \sqrt{\omega^2 - b^2(\eta)}\,d\eta - h_o t\right] + \frac{\pi}{4}\right\}}{\left\{2\pi\lambda\sqrt{\omega^2-b^2(x)}\,\sqrt{\omega^2-b^2(x_o)}\int_{x_<}^{x_>} \frac{b^2(\eta)}{(\omega^2-b^2(\eta))^{3/2}}\,d\eta\right\}^{1/2}}$$

Equations (30) and (35) give the asymptotic expansion of $u(t,x)$ parametrically with parameter ω.

In section (2) the asymptotic expansion of the Riemann Function will be obtained from the exact solution. The assumptions made to obtain the ray method expansion will be justified by the fact that the solutions agree.

(1.5) The Turned Rays and the Field Associated With Them

We assume that $b(x)$ is a monotonically decreasing function: $b'(x) < 0$. Then for each value of ω there exists at most one point, x_ω, for which $b^2(x_\omega) = \omega^2$. A point of this type will be called a turning point. In this section the ray method will be studied in the neighborhood of such points. (For a graph of a sample function, $b(x)$, see figure 2.)

We consider a ray emanating from a point $(0,x_o)$ for which the group speed, given by $\frac{k(x,\omega)}{\omega}$, is negative - i.e. x decreases as t increases. From (15) and (6), the equation of such a ray is given by

$$(36)\quad t(x) = \int_x^{x_o} \frac{h_o}{\sqrt{\omega^2-b^2(\eta)}}\,d\eta.^*$$

---

* It does not matter, for the discussion below, whether ω is the parameter (with x_o fixed) or σ is the parameter (with $x_o = \sigma$, $\omega^2 = h_o^2(s_o'(\sigma),\sigma)$).

At the turning point, $t = t(x_\omega)$. From (36) we see that $t(x_\omega)$ is independent of x and is a function only of the parameter (ω or σ). Also

$$(37) \qquad \left.\frac{dx}{dt}\right|_{x = x_\omega} = -\frac{\sqrt{\omega^2 - b^2(x_\omega)}}{h_o} = 0.$$

For $t > t(x_\omega)$, (9) must still be satisfied and therefore

$$(38) \qquad t(x) = t(x_\omega) + \int_{x_\omega}^{x} \frac{\omega}{k(\eta, \omega)}\, d\eta,$$

where $k(\eta,\omega)$ is again a solution of (6). Because t must be a real, increasing function of x, the integral on the right side of (38) must be real and positive. Therefore $k(\eta,\omega)$ must be real. This can only be true for $x > x_\omega$. Then for the integral to be positive $\frac{\omega}{k(\eta,\omega)}$ must be positive.

Hence $\dfrac{\omega}{k(\eta,\omega)} = \dfrac{h_o}{\sqrt{\omega^2-b^2(\eta)}}$.

With this information and (37), equation (38) can be rewritten as

$$(39) \qquad t(x) = \int_{x_\omega}^{x_o} + \int_{x_\omega}^{x} \frac{h_o}{\sqrt{\omega^2-b^2(\eta)}}\, d\eta; \quad t > t(x_\omega).$$

By differentiating with respect to x in (39), we see that $\frac{dx}{dt} > 0$ for $t > t(x_\omega)$. Those rays for which there exists a point, $x = x_\omega$, are described by (38) for $t < t(x_\omega)$ and by (39) for $t > t(x_\omega)$. The parts of the rays described by (36) and (39) will be called <u>primary rays</u> and <u>turned rays</u>, respectively. For rays emanating from (0,x) and moving to the right ($\frac{dx}{dt} > 0$), there is no turning

point. In this case, the entire ray is called a primary ray. These rays are shown in figure (1) at the end of this section. The contributions to the total solution associated with primary and turned rays will be denoted by u_P and u_T respectively.

By an analysis similar to that which was used to derive (39), the phase of u_T can be determined. The result is

$$(40) \qquad s^{\pm}(t,x) = s_o \pm \left\{ \int_{x_\omega}^{x_o} + \int_{x_\omega}^{x} \sqrt{\omega^2 - b^2(\eta)}\; d\eta - h_o t \right\}.$$

In the case of one dimensional or zero dimensional initial data for $s(t,x)$, the jacobians are given by (21) and (27) respectively. When there is a turning point those expressions are valid only for $t < t(x_\omega)$. Both contain an integral which diverges at $x = x_\omega$, but in each case $k(x,\omega)$ (which is zero at $x = x_\omega$) is a multiplicative factor of the integral. The limit of $j(t(x))$ as $x \to x_\omega$ exists and can be found by L'Hospital's rule.

To find the jacobian, $j(t(x))$, for $t > t(x_\omega)$, (39) must be implicitly differentiated with respect to the parameter (ω or σ). However, if one simply differentiates under the integral sign in (39), the resulting integral diverges. Instead, it is necessary to integrate by parts with respect to η in (39), before differentiating. When this procedure is carried out for the case of zero dimensional initial data (with ω as parameter), and one solves for $j_T(t(x)) = |\frac{\partial x}{\partial \omega}|$, the result is

$$(41)\quad j_T(t(x)) = \left| \frac{2\omega^2-b^2(x)}{\omega b(x)b'(x)} + \frac{\sqrt{\omega^2-b^2(x)}\ (2\omega^2-b^2(x_o))}{\omega\sqrt{\omega^2-b^2(x_o)}\ \ b(x_o)b'(x_o)} - \frac{\sqrt{\omega^2-b^2(x)}}{\omega} \left(\int_{x_\omega}^{x_o} + \int_{x_\omega}^{x} \right) \frac{2\omega^2-b^2(\eta)}{\sqrt{\omega^2-b^2(\eta)}} \left(\frac{1}{b(\eta)b'(\eta)} \right) d\eta \right|$$

Then, by the method of section (1.2.2), the amplitude is

$$(42)\quad z(t(x)) = \check{z}\left[\frac{j_o}{j_T(t(x))}\right]^{1/2}$$

with $j_T(t(x))$ and j_o given by (41) and (26) respectively. The field associated with the turned rays will be denoted by u_T. (39), (40) and (42) describe the rays, phase and amplitude, respectively, for u_T in the case of a zero dimensional initial manifold.

To find u_T for the Riemann function, we use (40)-(42) in (32) with u replaced by u_T($s_o^{\pm}$ and $z^{\pm}$ are given by (34).). The result is

$$(43)\quad u_T \sim \frac{\cos\left\{\lambda\left[\left(\int_{x_\omega}^{x_o} + \int_{x_\omega}^{x}\right)\sqrt{\omega^2-b^2(\eta)}\ d\eta - h_o t\right] + \frac{\pi}{4}\right\}}{\left[2\pi\lambda h_o\sqrt{\omega^2-b^2(x_o)}\quad j_T(t(x))\right]^{1/2}} .^*$$

* We note that $j_T(t,x)$, given by (41) might be zero for some (x,ω). Equivalently, there are caustics associated with one or more turned rays in (t,x) space. In this case, (43) gives u_T on the turned rays for times, t, less than the time at the caustic. By applying the discussion of phase change at a caustic in [2] and the results of section (2) below, we find that past the caustic, the phase term, $\frac{\pi}{4}$, appearing in (43) must be replaced by $-\frac{\pi}{4}$.

$u_T(t,x)$ is then given parametrically by (39) and (43).

To find the asymptotic expansion of the Riemann function at any point, (t,x), one must first determine all primary and turned rays passing through that point from (30) and (39). Then from (35) and (43) respectively the contributions u_P and u_T associated with each of these rays must be determined. The sum of all of these contributions is the asymptotic expansion of the Riemann function at that (t,x).

(1.6) An Initial-Boundary Value Problem - The Reflected Field.

We introduce a boundary at $x = x_B$ and the boundary condition

$$(44) \qquad u_x(t,x_B) - i\lambda\zeta u(t,x_B) = 0.$$

The total solution, to be determined for $x \geq x_B$, will be assumed to consist of three types of terms, denoted by u_P, u_T and u_R. u_T is that part of the solution associated with the turned rays and therefore non-zero only for $x > x_\omega \geq x_B$. u_P denotes the part of u associated with the primary rays, some of which reach the boundary. u_R will be associated with a new set of rays, to be called the <u>reflected rays</u>. For each primary ray "incident" on the boundary, a reflected ray will be "emitted." It is assumed that u_P satisfies the initial conditions and we therefore require that the reflected rays do not intersect the initial manifold; if they did, the total field at $(0,x)$ would be given by $u_P + u_R$ and would <u>not</u> satisfy the initial conditions.

For the purposes of this section, it suffices to assume that

$$(45) \qquad u = u_P + u_R.$$

In order to determine u_R by the ray method, it is necessary to find the values of ω, k, s and z for u_R at (t,x_B). The value of t is given by (15) or (30) with $x_< = x_B$. The value of b(x) at the boundary will be denoted by b_o.

It will be assumed that u_P and u_R are both of the forms given by (2). The subscripts P and R will be appended on the phase and amplitude of u_P and u_R respectively. Then from (45), to highest order in λ, (44) becomes

$$(46) \quad z_P(t,x_B)\exp[i\lambda s_P(t,x_B)]\left[\frac{\partial}{\partial x} s_P(t,x_B)-\zeta\right] + z_R(t,x_B)\exp[i\lambda s_R(t,x_B)]\left[\frac{\partial}{\partial x} s_R(t,x_B)-\zeta\right] = 0.$$

Since (46) must hold for all sufficiently large λ, it follows that

$$(47) \quad s_P(t,x_B) = s_R(t,x_B).$$

Differentiation of (47) with respect to t yields

$$(48) \qquad \frac{\partial}{\partial t} s_P(t,x_B) = \frac{\partial}{\partial t} s_R(t,x_B); \quad \omega_R = \omega_P = \pm h_{oP}.$$

It then follows from the dispersion relation, (6), that

$$(49) \qquad k_R(x,\omega_R) = \pm k_P(x,\omega_P).$$

If the signs were the same in (49), the reflected rays, determined as solutions of (9), would be the same as the primary rays and would therefore intersect the line (0,x). It has already been shown why this possibility must be rejected and therefore the lower sign must be chosen in (49). Then by using (5) it follows that

$$(50)\qquad \frac{\partial s_R(t,x_B)}{\partial x} = -\frac{\partial s_P(t,x_B)}{\partial x} .$$

In addition, by differentiating (31) with respect to x ($x = x_<$) and evaluating at $x = x_B$, we see that

$$(51)\qquad k^{\pm}_P(x_B,\ \omega) = \mp\sqrt{\omega^2-b_o^2}$$

and therefore

$$(52)\qquad k^{\pm}_R(x_B,\omega) = \pm\sqrt{\omega^2-b_o^2}\ ;\ k^{\pm}_R(x,\omega) = \pm\sqrt{\omega^2-b^2(x)}\ .^{*}$$

By substituting the proper value of k for $\frac{\partial s}{\partial x}$ in (46) and using (47), it follows that

$$(53)\qquad z^{\pm}_R(t,x_B) = R_{\pm}z^{\pm}_P(t,\ x_B)\ ;\ R_{\pm} = \frac{\pm\sqrt{\omega^2-b_o^2}+\zeta}{\pm\sqrt{\omega^2-b_o^2}-\zeta}$$

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* The special case $\omega^2 = b_o^2$ requires more careful investigation. For this case $k_P = k_R = 0$ and the group speed $(\frac{k}{\omega})$ is zero; the ray is tangent to the space time boundary. This case will not be considered here and therefore we assume that $\omega^2 > b_o^2$.

The reflected rays must again satisfy (9) with ω and k given by (48) and (52). The time at which a reflected ray is "emitted" is the same as the time at which the corresponding primary ray is incident on the boundary. We therefore find that

$$(54) \qquad t = \int_{x_B}^{x_o} + \int_{x_B}^{x} \frac{h_o}{\sqrt{\omega^2 - b^2(\eta)}} \, d\eta,$$

where x_o is fixed when ω is the parameter or $x_o = \sigma$, $\omega = \pm h_o(s_o'(\sigma), \sigma)$, when σ is the parameter. With the same consideration, by using (10) with k_R and ω_R determined above, we find that

$$(55) \qquad s_R^{\pm}(t,x) = s_o \pm \left\{ \int_{x_B}^{x_o} + \int_{x_B}^{x} \sqrt{\omega^2 - b^2(\eta)} \, d\eta - h_o t \right\} .$$

When σ is the parameter (i.e. the initial data for s is one dimensional), from (22):

$$(56) \quad z_P(t, x_B) = z_o \left[j(t(x_B)) \right]^{1/2} .$$

Then from (53) and (19) with $t_1 = t(x_B)$:

$$(57) \quad z_R^{\pm} = R_{\pm} \, z_o^{\pm} \left[j(t(x_B)) \right]^{-1/2} \left[\frac{j_R(t(x_B))}{j_R(t(x))} \right]^{1/2}$$

with $j_R(t(x))$ calculated from (54). It is easy to check that the jacobian for incident and reflected rays are the same at $x = x_B$, so that

$$(58) \qquad z_R^{\pm}(t(x)) = R_{\pm}\, z_o^{\pm}\left[j_R(t(x))\right]^{-\frac{1}{2}} .$$

For the zero dimensional manifold we obtain, in a similar fashion,

$$(59) \qquad z_R^{\pm}(t,x) = R_{\pm}\, \tilde{z}_o^{\pm}\left[\frac{j_o}{j_R(t(x))}\right]^{\frac{1}{2}} .$$

Again, $j_R(t(x))$ must be calculated from (54).

In the case of the Riemann function for the bounded domain, $x \geq x_B$, we find that

$$(60) \qquad u_R \sim \sum_{\pm} \frac{R_{\pm} \exp\left\{\pm i\lambda\left[\int_{x_B}^{x_o} + \int_{x_B}^{x} \sqrt{\omega^2-b^2}\, d\eta - h_o t\right] \pm \frac{i\pi}{4}\right\}}{\left[8\pi\lambda\, h_o\sqrt{\omega^2-b^2(x_o)}\;\; j_R(t(x))\right]^{\frac{1}{2}}}$$

where

$$(61) \qquad j_R(t(x)) = \frac{\sqrt{\omega^2-b^2(x)}}{h_o} \int_{x_B}^{x_o} + \int_{x_B}^{x} \frac{b^2(\eta)}{(\omega^2-b^2(\eta))^{3/2}}\, d\eta$$

as calculated from (54). $u_R(t,x)$ is then given parametrically by (54) and (59). From (61) one can see that there are no caustics associated with the reflected field for the Riemann function.

To find the asymptotic expansion of the Riemann function for the bounded region, $x \geq x_B$, we must append the discussion at the end of section (1.5): in addition to the primary and turned rays passing through a point (t,x), one must determine the reflect rays and add to the solution the contributions, u_R, associated with these rays. (Reflected rays are shown in figure (1).)

For any other problem, the expansion by the ray method is interpreted in exactly the same way: the solution for any (t,x) is given by a sum of contributions associated with all the rays passing through the point.

The asymptotic expansion of the Riemann function will be obtained in the next section by asymptotically expanding the exact solution. It will be seen that the two solutions agree. In subsequent sections, other types of initial data and sources will be considered, and, in each case, the solutions will agree.

2. Introduction to the Asymptotic Expansion of the Exact Solution

The equation to be studied in this section is

$$(1)\quad u_{xx} - u_{tt} - \lambda^2 b^2(x) u = f(t,x;\lambda)$$

with initial conditions

(2) $u(0,x) = u_o(x;\lambda)$, $u_t(0,x) = u_1(x;\lambda)$

and boundary condition given by (1.44). It is again assumed that $b(x)$ is monotonically decreasing, $b'(x) < 0$; $b(x_B) = b_o$ and $\lim_{x\to\infty} b(x) = b_1$.

To solve (1), we separate variables by taking the Fourier transform of $u(t,x)$ with respect to t. This transform will be called $v(x,\omega)$. It will be shown below that this function must satisfy an ordinary differential equation in x. When $v(x,\omega)$ is known, its inverse Fourier transform will yield the solution, $u(t,x)$.

In general, only the asymptotic expansion of $v(x,\omega)$ for large λ can be determined and not the function itself. The inverse transform of this expansion is an integral which can be expanded by standard methods of asymptotic expansions of integrals with a large parameter.

It is assumed that $u(t,x) = 0$ for $t < 0$. We also assume that $u(t,x)e^{-ct}$ is absolutely integrable on $0 \leqq t < \infty$, for all $c > 0$ and that the absolute integral is uniformly bounded, with respect to x. The Fourier transform, $v(x,\omega)$, is given by

$$(3)\quad v(x,\omega) = \int_0^\infty u(t,x)\exp[i\lambda\omega t]dt, \quad \operatorname{Im}\omega > 0.$$

In this case, the Fourier Inversion Theorem can be applied to yield

$$(4)\quad u(t,x) = \frac{\lambda}{2\pi}\int_\Gamma v(x,\omega)\exp[-i\lambda\omega t]d\omega.$$

The path of integration, Γ, is the real ω axis, except in the neighborhood of the singularities of $v(x,\omega)$; the path is deformed above these singularities. Under suitable conditions on v we may rewrite (4) as

$$(5)\quad u(t,x) = \frac{\lambda}{2\pi}\sum_{\pm}\int_0^\infty v(x,\pm\omega)\exp[\mp i\lambda\omega t]d\omega$$

with the understanding that (5) really has meaning as a limiting case of (4) when the deformations around the singularities of $v(x,\pm\omega)$ shrink onto the

singular points. Of course, (5) would be incorrect if $v(x,\pm\omega)$ had non-integrable singularities on the axis. In this case one must refer to (4) as a defining equation for $u(t,x)$.

The ordinary differential equation satisfied by $v(x,\omega)$ is obtained by taking the Fourier transform of each term in (1). The resulting equation is

$$(6)\quad v''(x,\omega) + \lambda^2(\omega^2-b^2(x))\, v(x,\omega) = r(x,\omega;\lambda);$$

where

$$(7)\quad r(x,\omega;\lambda) = i\lambda\omega u_o(x;\lambda)-u_1(x;\lambda) + \int_0^\infty f(t,x;\lambda)\exp[i\lambda\omega t]dt$$

and (') denotes differentiation with respect to x. When computing the transform of u_{tt}, it is necessary to integrate by parts twice with respect to t. Each integration introduces an endpoint contribution at $t = 0$. This accounts for the appearance of the initial data in $r(x,\omega;\lambda)$.

Transformation of the boundary condition, (1.44), yields

$$(8)\quad v'(x_B,\omega)-i\lambda\zeta v(x_B,\omega) = 0.$$

The following "radiation condition" can be derived:

$$(9)\quad \lim_{x\to\infty} v(x,\omega) = 0, \ \mathrm{Im}\ \omega > 0.$$

The assumption that the integral of $|u(t,x)|e^{-ct}$ is uniformly bounded in x, is used to derive (9). In addition, it is necessary to impose the restriction that u_o, u_1 and f have compact support in x. As a consequence of this restriction, $r(x,\omega;\lambda)$ has compact

support in x. This will be assumed throughout the discussion.

2.1 The General Technique of Solution of the Time Reduced Equation.

We consider, first, the homogeneous form of (6):

$$(10) \quad v''(x,\omega) + \lambda^2(\omega^2 - b^2(x))v(x,\omega) = 0.$$

If $v_1(x,\omega)$ and $v_2(x,\omega)$ are linearly independent solutions of the homogeneous equation which satisfy the radiation condition and the boundary condition respectively, then the solution to (6), (7), (8), (9) is given by

$$(11) \quad v(x,\omega) = -\frac{1}{W(v_1,v_2)}\left[v_1(x,\omega)\int_{x_B}^{x} v_2(\xi,\omega)r(\xi,\omega;\lambda)d\xi \right.$$

$$\left. + v_2(x,\omega)\int_{x}^{\infty} v_1(\xi,\omega)r(\xi,\omega;\lambda)d\xi\right].$$

The Wronskian, $W(v_1,v_2) = v_1v_2' - v_2v_1'$, is independent of x.

We define Region I: $\omega^2 > b_o^2$, $x \geq x_B$. (See figure (2).) The functions v_1 and v_2 are obtained in this region by formally substituting a solution of the form $a(x,\omega)\exp[i\lambda s(x,\omega)]$ into (10), [6]. For $b_o^2 > \omega^2 > b_1^2$, there exists a point $x = x_\omega$ such that $b(x_\omega) = \omega$. This is called a turning point [6]. We define Region II: $b_o^2 > \omega^2 > b_1^2$, $x \geq x_\omega$. (See figure (2).) For this region, v_1 and v_2 are obtained by the WKB method [6,7,8]. It can be shown that $v(x,\omega)$ is exponentially small for

real ω except in Regions I and II. (The region were v is exponentially small is shaded in figure 2.) For $\omega = b_o$, $x_\omega = x_B$. Since the solution obtained by the WKB method is not valid at the turning point the boundary condition cannot be satisfied for $\omega = b_o$. This difficulty is discussed in section (9).

In order to use (5) to determine $u(t,x)$, it is necessary to find $v(x,\pm\,\omega)$. Equation (11) can be used for this purpose if $v_1(x,\pm\,\omega)$ and $v_2(x,\pm\omega)$ are known. The expansions of $v_1(x,\pm\omega)$ and $v_2(x,\pm\omega)$ will be given for regions I and II in sections (2.2) and (2.3), respectively.

(2.2) $v_1(x,\pm\omega)$ and $v_2(x,\pm\omega)$ in Region I: $\omega > b_o$, $x \geqq x_B$

In Appendix I, Part 1, it is shown that solutions to (10) which satisfy the radiation condition and boundary condition, respectively, are given for $\omega > b_o$ by

$$(12)\quad v_1(x,\pm\omega) \sim (\omega^2-b^2(x))^{-\frac{1}{4}} \exp\left[\pm i\lambda\int_{x_B}^{x} \sqrt{\omega^2-b^2(\eta)}\;d\eta\right]$$

$$(13)\quad v_2(x,\pm\,\omega) \sim (\omega^2-b^2(x))^{-\frac{1}{4}} \left\{\exp\left[i\lambda\int_{x_B}^{x} \sqrt{\omega^2-b^2(\eta)}\;d\eta + i\gamma\right]\right.$$

$$\left. -\exp\left[-i\lambda\int_{x_B}^{x} \sqrt{\omega^2-b^2(\eta)}\;d\eta - i\gamma\right]\right\};\ \cot\gamma = \frac{i\zeta}{\sqrt{\omega^2-b_o^2}}\ .$$

The Wronskian is

(14) $W(v_1, v_2) \sim 2i\lambda e^{\mp i\gamma}$.

Since all results are asymptotic expansions, it would be tedious to continually repeat such expressions as "the asymptotic expansion of the solution is ...". Instead, such expressions are simply stated as "the solution is ...". The sign of equality or asymptotic equality will convey the proper meaning. This convention is adhered to in the discussion directly above and in all that follows.

The solutions given in (12) and (13) could each be multiplied by a constant and still be solutions, since they are determined by one condition and satisfy a second order equation. These constants would both be factors of W and would have no effect on (11). Therefore (12) and (13) are called <u>the</u> solutions despite their non-uniqueness.

<u>(2.3) $v_1(x, \pm\omega)$ and $v_2(x, \pm\omega)$ in Region II: $b_o > \omega > b(x)$, $x > x_\omega$</u>

In Appendix I, Part 2, the solutions which satisfy the radiation condition and the boundary condition, respectively, are determined by using the WKB method. They are

(15) $$v_1(x, \pm\omega) \sim \frac{(\omega^2 - b^2(x))^{-\frac{1}{4}}}{2i\lambda} \exp\left[\pm i\lambda \int_{x_\omega}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta\right],$$

$$(16)\quad v_2(x,\pm\omega) \sim \frac{(\omega^2-b^2(x))^{-\frac{1}{4}}}{2i\lambda}\, e^{\mp\frac{i\pi}{4}} \left\{\exp\left[i\lambda\int_{x_\omega}^{x}\sqrt{\omega^2-b^2(\eta)}\,d\eta + \frac{i\pi}{4}\right]\right.$$

$$\left. -\exp\left[-i\lambda\int_{x_\omega}^{x}\sqrt{\omega^2-b^2(\eta)}\,d\eta - \frac{i\pi}{4}\right]\right\}.$$

The Wronskian, $W = 1$.

(2.4) The Asymptotic Expansion of the Riemann Function

As an example, the Riemann function for (1), on the domain $x \geqq x_B$, will be determined. For this function $f(t,x;\lambda) = 0$, $u_o(x;\lambda) = 0$ and $u_1(x;\lambda) = \delta(x-x_o)$. From (8), for this problem,

$$(17)\quad r(x,\omega;\lambda) = -\delta(x-x_o).$$

Consequently, equation (11) reduces to

$$(18)\quad v(x,\omega) = \frac{v_1(x_>,\omega)v_2(x_<,\omega)}{W(v_1,v_2)}; \quad x_> = \max(x,x_o),\ x_< = \min(x,x_o).$$

For $\omega > b_o$, using the results of section (2.2) in (18) we obtain

$$(19)\quad v(x,\pm\omega)\sim \mp(2i\lambda)^{-1}(\omega^2-b^2(x))^{-\frac{1}{4}}(\omega^2-b^2(x_o))^{-\frac{1}{4}}$$

$$\left\{R_\pm \exp\left[\pm i\lambda\left(\int_{x_B}^{x_o}+\int_{x_B}^{x}\right)\sqrt{\omega^2-b^2(\eta)}\,d\eta\right]+\exp\left[\pm i\lambda\int_{x_<}^{x_>}\sqrt{\omega^2-b^2(\eta)}\,d\eta\right]\right\};$$

$$R_\pm = -e^{\pm 2i\gamma} = \frac{\pm\sqrt{\omega^2-b_o^2}+\zeta}{\pm\sqrt{\omega^2-b_o^2}-\zeta}$$

The contributions to u arising from (19) can be calculated by the method of stationary phase [6]. The contributions from the terms involving the reflections coefficients $R_\pm$ will be called u_R; the contributions from the other terms will be called $u_P^{(1)}$. By using the method of stationary phase we find that u_R is given parametrically with parameter ω by the equations

$$(20)\quad u_R\sim\sum_\pm \frac{R_\pm \exp\left[\pm i\lambda\left\{\left(\int_{x_B}^{x_o}+\int_{x_B}^{x}\right)\sqrt{\omega^2-b^2(\eta)}\,d\eta-\omega t\right\}\pm\frac{i\pi}{4}\right]}{\left\{8\pi\lambda\sqrt{\omega^2-b^2(x)}\sqrt{\omega^2-b^2(x_o)}\left(\int_{x_B}^{x_o}+\int_{x_B}^{x}\right)\frac{b^2(\eta)}{(\omega^2-b^2(\eta))^{3/2}}\,d\eta\right\}^{1/2}},$$

$$(21)\quad t=\left(\int_{x_B}^{x_o}+\int_{x_B}^{x}\right)\frac{\omega}{\sqrt{\omega^2-b^2(\eta)}}\,d\eta;\ \omega>b_o\,.$$

where the latter equation is just a consequence of the requirement that the phase be stationary. Similarly we find that $u_P^{(1)}$ is given parametrically by

$$(22)\quad u_P^{(1)} \sim \frac{\cos\left[\lambda\left\{\int_{x_<}^{x_>} \sqrt{\omega^2-b^2(\eta)}\, d\eta - \omega t\right\} + \frac{\pi}{4}\right]}{\left\{2\pi\lambda\sqrt{\omega^2-b^2(x)}\ \sqrt{\omega^2-b^2(x_o)}\ \int_{x_<}^{x_>} \frac{b^2(\eta)}{(\omega^2-b^2(\eta))^{3/2}}\, d\eta\right\}^{1/2}} ;$$

$$(23)\quad t = \int_{x_<}^{x_>} \frac{\omega}{\sqrt{\omega^2-b^2(\eta)}}\, d\eta ; \qquad \omega > b_o .$$

For the range $b_o > \omega > b_1$, the functions given in section (2.3) are substituted into (18) to find $v(x,\pm\omega)$. The

$$(24)\quad v(x,\pm\omega) \sim \mp \frac{\mp i \exp\left[\pm i\lambda\left\{\int_{x_\omega}^{x_o} + \int_{x_\omega}^{x} \sqrt{\omega^2-b^2(\eta)}\, d\eta\right\}\right] + \exp\left[\pm i\lambda \int_{x_<}^{x_>} \sqrt{\omega^2-b^2(\eta)}\, d\eta\right]}{2i\lambda(\omega^2-b^2(x))^{1/4}\ (\omega^2-b^2(x_o))^{1/4}} ,$$

The second term of (24) is exactly the same as the second term of (19), and therefore the contribution to $u(t,x)$ from this term is given by (22) and (23) with one change: the ω interval, for which (24) is a valid representation of v, is the interval, $b(x_<) < \omega < b_o$. Therefore the condition which appears in (23), namely, $\omega > b_o$, must be replaced by $b_o > \omega > b(x_<)$. A function $u_P(t,x)$ can be introduced to include (22) and the contribution to $u(t,x)$ coming from the second term of (24). This function is defined parametrically by

$$(25)\quad u_P = u_P^{(1)}; \quad t = \int_{x_<}^{x_>} \frac{\omega}{\sqrt{\omega^2 - b^2(\eta)}}\, d\eta; \quad b(x_<) < \omega.$$

If the contributions to $u(t,x)$ from the first term of (24) are called $u_T(t,x)$, then

$$(26)\quad u_T(t,x) \sim \sum_{\pm} \mp \frac{1}{4\pi i} \int_{b(x_<)}^{b_o} \frac{\mp i \exp\left[\pm i\lambda \left(\int_{x_\omega}^{x_o} + \int_{x_\omega}^{x}\right) \sqrt{\omega^2 - b^2(\eta)}\, d\eta - \omega t\right]}{(\omega^2 - b^2(x))^{1/4}(\omega^2 - b^2(x_o))^{1/4}}\, d\omega.$$

We again apply the method of stationary phase to the integral. It should be noted that in order to find the second derivative of the phase with respect to ω, it is necessary to integrate by parts with respect to η before differentiating under the integral sign. If this is not done the factor $(\omega^2 - b^2(\eta))^{3/2}$ will appear in the denominator of the integrand and cause the integral to diverge at the lower limit of integration, $\eta = x_\omega$. The resulting expression for $u_T(t,x)$ is given by the parametric representation

$$(27)\quad u_T \sim \frac{\cos\left[\lambda\left\{\left(\int_{x_\omega}^{x_o} + \int_{x_\omega}^{x}\right) \sqrt{\omega^2 - b^2(\eta)}\, d\eta - \omega t\right\} + \frac{\pi}{4}\right]}{\left\{2\pi\lambda \sqrt{\omega^2 - b^2(x)}\ \sqrt{\omega^2 - b^2(x_o)}\ \left|\frac{d^2\varphi}{d\omega^2}\right|\right\}^{1/2}}$$

$$(28)\quad t = \left(\int_{x}^{x_o} + \int_{x_\omega}^{x}\right) \frac{\omega}{\sqrt{\omega^2 - b^2(\eta)}}\, d\eta; \quad b_o > \omega > b(x),$$

where

$$(29)\qquad \frac{d^2\varphi}{d\omega^2} = -\frac{2\omega^2-b^2(x)}{b(x)b'(x)\sqrt{\omega^2-b^2(x)}} - \frac{2\omega^2-b^2(x_o)}{b(x_o)b'(x_o)\sqrt{\omega^2-b^2(x_o)}}$$

$$+\int_{x_\omega}^{x_o} + \int_{x_\omega}^{x} \left[\frac{1}{b(\eta)b'(\eta)}\right]' \frac{2\omega^2-b^2(\eta)}{\sqrt{\omega^2-b^2(\eta)}}\, d\eta \quad .$$

This result is valid only up to the first zero of (29), or equivalently up to the first caustic. Past the caustic the phase factor $\frac{\pi}{4}$ in (27) must be replaced by $-\frac{\pi}{4}$.

(2.5) Comparison of the Results

The functions, u_P, u_T and u_R, given by equations (1.35), (1.43) and (1.60) agree with the corresponding results of section 2, given by equations (2.25), (2.27) and (2.20).* The stationary phase conditions

* One must note that $\omega > 0$ in the method of section (2). As a parameter, it plays the same role as h_o does in the ray method.

of section 2 are seen to be exactly the equations of the various types of rays in section 1.

To find the Riemann function by the Ray Method, it was assumed that all primary rays emanated from the point x_o. This assumption is now justified by the agreement of the results.

(3) Rapidly Varying Initial Data - Ray Method

If equation (1.1) were originally given without a large parameter λ, "stretching" the variables x and t by this factor would lead to the given form of (1.1) and to initial data of the form

(1) $u(0,x) = u_o[\lambda(x-x_o)]$, $u_t(0,x) = \lambda u_1[\lambda(x-x_o)]$.

This will be called rapidly varying initial data [2]. We assume that $u_o(\xi)$ and $u_1(\xi)$ have compact support. Then the support of the initial data shrinks to the point $x = x_o$ as $\lambda \to \infty$. As in the case of the Riemann function, it is reasonable to assume that the rays emanate from the point $(0,x_o)$. Then the results of subsections (1.2.2, 1.3.2) apply. The rays, phase and amplitude are the same as for the Riemann function, except that the constants, $s_o^{\pm}$ in (1.31) and $\tilde{z}^{\pm}$ in (1.27) must again be calculated by the indirect method for the initial data (1). The expansion of the exact solution for this problem with b = constant is obtained in [2] and yields the results

(2) $s_o^{\pm} = 0$, $\tilde{z}^{\pm} = \left(\dfrac{h_o^3}{8\pi\lambda b^2(x_o)}\right)^{1/2} a_{\pm}(k_o)e^{\mp\frac{i\pi}{4}}$;

$k_o = \pm\,[\mathrm{sgn}(x-x_o)]\sqrt{\omega^2-b^2(x_o)}$;

(3) $a_{\pm}(k_o) = \displaystyle\int_{-\infty}^{\infty} \left[u_o(\xi) \pm \frac{i}{h_o}\, u_1(\xi)\right] e^{-ik_o\xi}d\xi$; $h_o = \sqrt{k_o^2 + b^2(x_o)}$.

We substitute (3), (1.26) and (1.28) in (1.27) and find that

$$(4)\qquad z^{\pm} = \frac{h_o a_{\pm}\left(\pm\,[\mathrm{sgn}(x-x_o)]\sqrt{\omega^2-b^2(x_o)}\right)e^{\mp\frac{i\pi}{4}}}{\left(8\pi\lambda\sqrt{\omega^2-b^2(x_o)}\,\sqrt{\omega^2-b^2(x)}\displaystyle\int_{x_<}^{x_>}\frac{b^2(\eta)}{(\omega^2-b^2(\eta))^{3/2}}\,d\eta\right)^{1/2}}\,.$$

From (1.32), $u_P(t,x)$ is given parametrically, with parameter ω, by the equation of the rays, (1.30), and

$$(5)\qquad u_P \sim \sum_{\pm} z^{\pm}\exp\left[\pm\, i\lambda\left\{\int_{x_<}^{x_>}\sqrt{\omega^2-b^2(\eta)}\;d\eta - h_o t\right\}\mp\frac{i\pi}{4}\right].$$

When (3) is compared with (1.34), one observes that the former result is just the latter multiplied by $\mp\, ih_o a_{\pm}\left(\pm\,[\mathrm{sgn}(x-x_o)]\sqrt{\omega^2-b^2(x_o)}\right)$. For rays in the direction of decreasing x this factor reduces to

$$(6)\qquad p_{\pm} = \mp\, ih_o a_{\pm}\left(\mp\sqrt{\omega^2-\,b^2(x_o)}\right).$$

If $p_{\pm}$ is introduced as a factor in the summations, $\sum_{\pm}$, expressing u_T and u_R for the Riemann function, the resulting equations are the corresponding results for the present problem. To find u_T by this method, (1.43) must first be rewritten as a sum, $\sum_{\pm}$. The result is

$$(7)\quad u_T \sim \sum_{\pm} h_o a_{\pm}\left[\mp\sqrt{\omega^2-b^2(x_o)}\right] \frac{\exp\left[\pm i\lambda\left\{\int_{x_\omega}^{x_o}+\int_{x_\omega}^{x}\sqrt{\omega^2-b^2(\eta)}\,d\eta-h_o t\right\}\mp\frac{i\pi}{4}\right]}{\left[8\pi\lambda\, h_o\sqrt{\omega^2-b^2(x_o)}\quad j_T(t(x))\right]^{1/2}},$$

where j_T is given by (1.41). The parametric representation of u_R in (1.60) is of the form $u_R^+ + u_R^-$. Using these functions the corresponding result for the present problem is

$$(8)\quad u_R \sim \sum_{\pm} p_{\pm}\, u_R^{\pm}\,.$$

In (9) $\omega > b_o$, $x \geq x_B$ and in (8), $\omega < b_o$, $x > x_\omega$.

(4) Rapidly Varying Initial Data - Expansion of the Exact Solution

The problem introduced in section (3) will be treated by the method of section (2). To do this, we calculate the right side of (2.1) by using equations (2.7), (2.2) and (3.1). The result is

$$(1)\quad r\,(x, \pm\,\omega; \lambda) = \pm\, i\lambda\omega\left[u_o(\lambda(x-x_o)) \pm \frac{i}{\omega}\, u_1(\lambda(x_1-x_o))\right]\,.$$

For $u_o(x)$ and $u_1(x)$ having compact support in x, the support of $r(x, \pm\,\omega; \lambda)$ shrinks to x_o as $\lambda \to \infty$. In this case, for λ sufficiently large and $x \neq x_o$, only one integral in (2.11) is non-zero. Therefore

$$(2)\quad v(x,\pm\omega) = -\frac{1}{W(v_1,v_2)}\int_{-\infty}^{\infty} v_1(x_1,\pm\omega)v_2(x_2,\pm\omega)r(\xi,\pm\omega;\lambda)d\xi,$$

where

$$(3)\quad \begin{array}{ll} x_2 = x,\ \ x_1 = \xi & \text{for } x < x_o \\ x_1 = x,\ x_2 = \xi & \text{for } x > x_o\ . \end{array}$$

Using (1), (2) and (3), $v(x,\pm\omega)$ can be determined for the regions I and II defined in section (2).

(4.1) The Expansion of $v(x,\pm\omega)$ in Region I: $\omega > b_o$, $x \geq x_B$

We use (2.12), (2.13), (2.14), (2) and (3) to write v as a sum of two terms, $v = v_P^{(1)} + v_R$, where

$$(4)\quad v_R(x,\pm\omega) \sim \pm\frac{R_\pm \exp\left[\pm i\lambda\left(\int_{x_B}^{x_o} + \int_{x_B}^{x}\right)\sqrt{\omega^2-b^2(\eta)}\ d\eta\right]}{2i\lambda(\omega^2-b^2(x))^{1/4}} \int_{-\infty}^{\infty}\frac{r(\xi,\pm\omega;\lambda)\exp\left[\pm i\lambda\int_{x_o}^{\xi}\sqrt{\omega^2-b^2(\eta)}\ d\eta\right]}{(\omega^2-b^2(\xi))^{1/4}}\,d\xi\ ;$$

$$(5)\quad v_P^{(1)} \sim \pm\frac{1}{2i\lambda(\omega^2-b^2(x))^{1/4}}\int_{-\infty}^{\infty}\frac{r(\xi,\pm\omega;\lambda)\exp\left[\pm i\lambda\int_{x_2}^{x_1}\sqrt{\omega^2-b^2(\eta)}\ d\eta\right]}{(\omega^2-b^2(\xi))^{1/4}}\,d\xi.$$

In Appendix II, it is shown rigorously that the integral with respect to ξ, appearing in (4), can be expanded asymptotically. Heuristically, we can arrive at the same result by using the fact that the support of r shrinks to x_o in order to replace $(\omega^2-b^2(\xi))^{1/4}$ by $(\omega^2-b^2(x_o))^{1/4}$ and $\int_{x_o}^{\xi} \sqrt{\omega^2-b^2(\eta)}\; d\eta$ by $\sqrt{\omega^2-b^2(x_o)}\,(\xi-x_o)$. The resulting integral has the advantage of being simply a Fourier transform of the initial data. v_R can then be rewritten as

$$(6)\qquad v_R(x, \pm\,\omega) \sim \frac{R_{\pm}\omega a_{\pm}\left[\mp\sqrt{\omega^2-b^2(x_o)}\right]\exp\left[\pm i\lambda \int_{x_B}^{x_o} + \int_{x_B}^{x} \sqrt{\omega^2-b^2(\eta)}\;\; d\eta\right]}{2\lambda(\omega^2-b^2(x))^{1/4}(\omega^2-b^2(x_o))^{1/4}},$$

where $a_{\pm}(k_o)$ is given by (3.3).

Equation (5) can be simplified in a similar manner if one first notes that

$$(7)\quad x_1-x_2 = x_> - x_< - [\mathrm{sgn}(x-x_o)](\xi-x_o);\; x_> = \max(x,x_o),\; x_< = \min(x,x_o).$$

The resulting expression for $v_P^{(1)}$ is

$$(8)\quad v_P^{(1)}(x,\pm\,\omega) \sim \frac{\omega a_{\pm}\left[\pm\, [\mathrm{sgn}(x-x_o)]\sqrt{\omega^2-b^2(x_o)}\,\right]\exp\left[\pm i\lambda \int_{x_<}^{x_>} \sqrt{\omega^2-b^2(\eta)}\; d\eta\right]}{2\lambda(\omega^2-b^2(x))^{1/4}(\omega^2-b^2(x_o))^{1/4}}.$$

(4.2) The Expansion of $v(x,\pm\omega)$ in Region II: $b_o > \omega > b(x_<)$, $x > x_\omega$.

We use (2.15) and (2.16) to find v in Region II. The results are similar to those above except that x_B is replaced by x_ω and $R_\pm$ is replaced by $\mp i$. If v is written as a sum, $v = v_P^{(2)} + v_T$, then

$$\text{(11)}\quad v_T(x,\pm\omega) \sim \frac{\mp i\ \omega a_\pm\left[\mp\sqrt{\omega^2-b^2(x_o)}\right] \exp\left[\pm i\lambda \int_{x_\omega}^{x_o} + \int_{x_\omega}^{x} \sqrt{\omega^2-b^2(\eta)}\, d\eta\right]}{2\lambda(\omega^2-b^2(x))^{1/4}\ (\omega^2-b^2(x_o))^{1/4}},$$

$$\text{(12)}\quad v_P^{(2)}(x,\pm\omega) = v_P^{(1)}(x,\pm\omega).$$

(4.3) The Solution

To find u_P, u_T, and u_R, the inverse transforms of (7), (10), (11) and (12) must be expanded asymptotically. This process is simplified by using the results already obtained for the Riemann function. If the first term of (2.19) is compared with (7), we find that the latter could be obtained by multiplying the former by $p_\pm$, where

$p_\pm = \mp i\omega a_\pm\left(\mp\sqrt{\omega^2-b^2(x_o)}\right)$. This factor is independent of λ. When using the method of stationary phase to expand the inverse transform of $v_R(x,\pm\omega)$, the factor is simply evaluated at the stationary point. Therefore, if this factor is introduced into the sum, $\sum_\pm$, in (2.20), the result will be u_R for this problem. This is exactly the factor given by (3.7), and our present multiplication technique is the same as was used in section (3) to

obtain u_R from the ray method results for the Riemann function. Since the results for u_R in the problem for the Riemann function agree, the same must be true in this problem. This comparison technique also allows us to conclude that the results for u_T and u_P must agree with the results of section (3).

(5) Oscillatory Initial Data - Ray Method

We consider now an initial-boundary value problem for (1.1) where the initial data is designed to fit the form of (1.2); that is

$$(1)\quad u(x,0) \sim z_o(x)\exp[i\lambda s_o(x)],\ u_t(x,0) \sim -i\lambda h_o(s_o'(x),\ x)z_o(x)\exp[i\lambda s_o(x)];$$

$$h_o(k,x) = \sqrt{k^2 + b^2(x)}\ .$$

The functions, s_o and z_o, are given; z_o is assumed to have compact support bounded away from $x=x_B$. Data of this form will be called <u>oscillatory initial data</u> [9]. The boundary condition is again given by (1.44). (For problems where $u(x,0) \sim z_o \exp[i\lambda s_o]$, $u_t(x,0) \sim z_1 \exp[i\lambda s_1]$ with z_o, z_1, s_o and s_1 arbitrary, u can be found by a superposition of solutions of problems with data of the form of (1). The method to be used in this more general case is described in [2].)

For a solution of the form of (1.2), equation (1) yields the values of s and z at t = 0:

(2) $s(0,\sigma) = s_o(\sigma)$; $z(0,\sigma) = z_o(\sigma)$.

This equation gives $s(0,x)$ initially on the one dimensional manifold, $t = 0$, so that we may use the results of sections (1.2.1) and (1.3.1) to obtain $u_p(t,x)$. The resulting parametric representation (with parameter σ) is

$$(3)\quad u_p \sim z_o(\sigma)[j(t(x))]^{-1/2} \exp\left[i\lambda\left\{s_o(\sigma) + \int_{x_<}^{x_>} \sqrt{\omega^2-b^2(\eta)}\, d\eta - h_o t\right\}\right];$$

$$(4)\quad t = \int_{x_<}^{x_>} \frac{h_o}{\sqrt{\omega^2-b^2(\eta)}}\, d\eta, \quad x_< = \min(x,\sigma),\ x_> = \max(x,\sigma).$$

$j(t(x))$ is given by (1.21). To insure that there are no caustics we assume that $s_o'' > 0$. The solution will contain the features of reflected and turned rays if it is also assumed that $s_o'(\sigma) < 0$. In this case, from (1.9), (1.13) and (1.14)

$$(5)\quad \left.\frac{dx}{dt}\right|_{(0,\sigma)} = \frac{[\operatorname{sgn} s_o'(\sigma)]\sqrt{\omega^2-b^2(\sigma)}}{\omega} < 0;\ \omega = h_o(s_o'(\sigma),\sigma) = \sqrt{s_o'^2(\sigma)+ b^2(\sigma)}.$$

For $\omega^2 < b_o^2$, each primary ray gives rise to a turned ray. From (1.39), the equation of the turned rays is

$$(6)\quad t(x) = \int_{x_\omega}^{\sigma} + \int_{x_\omega}^{x} \frac{h_o}{\sqrt{\omega^2 - b^2(\eta)}}\, d\eta;\ t > t(x_\omega).$$

The phase on the turned rays is given by (1.40), where the choice of sign, ($\pm$), refers to sgn ω. For the present case sgn $\omega = 1$. The amplitude on the turned rays is given by (1.22) with the jacobian, $j_T(t(x))$ calculated from (6) by the method described in section (1.5). The result is

$$(7)\quad j_T(t(x)) = \Bigg| \frac{\sqrt{\omega^2 - b^2(x)}}{\sqrt{\omega^2 - b^2(\sigma)}} - \frac{\sqrt{\omega^2 - b^2(x)}}{\omega^2} \left(s_o'(\sigma) s_o''(\sigma) + b(\sigma) b'(\sigma)\right)$$

$$\cdot \left\{ \frac{2\omega^2 - b^2(\sigma)}{b(\sigma) b'(\sigma) \sqrt{\omega^2 - b^2(\sigma)}} + \frac{2\omega^2 - b^2(x)}{b(x) b'(x) \sqrt{\omega^2 - b^2(x)}} \right.$$

$$\left. + \int_{x_\omega}^{\sigma} + \int_{x_\omega}^{x} \left(\frac{1}{b(\eta) b'(\eta)} \right)' \frac{2\omega^2 - b^2(\eta)}{\sqrt{\omega^2 - b^2(\eta)}}\, d\eta \right\} \Bigg| .$$

With these results, $u_T(t,x)$ is given parametrically by (6) and

$$(8)\quad u_T \sim z_o(\sigma)\left[j_T(t(x))\right]^{-1/2} \exp\left[i\lambda\left\{s_o(\sigma) + \int_{x_\omega}^{\sigma} + \int_{x_\omega}^{x} \sqrt{\omega^2-b^2(\eta)}\, d\eta - h_o t\right\}\right]$$

The reflected rays are given by (1.54) with $x_o = \sigma$:

$$(9)\quad t = \int_{x_B}^{\sigma} + \int_{x_B}^{x} \frac{h_o}{\sqrt{\omega^2-b^2(\eta)}}\, d\eta.$$

The phase for u_R is obtained from (1.55) with sgn $\omega = 1$. $z(t(x))$ is given by (1.58). The jacobian, $j_R(t(x)) = \left|\frac{\partial x}{\partial \sigma}\right|$, must be calculated from (9). The result is

$$(10)\quad j_R(t(x)) = \left| \frac{\sqrt{\omega^2-b^2(x)}}{\sqrt{\omega^2-b^2(\sigma)}} - \frac{\sqrt{\omega^2-b^2(x)}}{\omega^2}\left(s_o'(\sigma)s_o''(\sigma) + b(\sigma)b'(\sigma)\right) \cdot \int_{x_B}^{\sigma} + \int_{x_B}^{x} \frac{b^2(\eta)}{(\omega^2-b^2(\eta))^{3/2}}\, d\eta \right|.$$

These results allow us to express $u_R(t(x))$ parametrically by (9) and

$$(11)\quad u_R \sim R_+ z_o(\sigma)\left[j_R(t(x))\right]^{-1/2} \exp\left[i\lambda\left\{s_o(\sigma) + \int_{x_B}^{\sigma} + \int_{x_B}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta - h_o t\right\}\right]$$

with R_+ given by (1.53).

To find $u(t,x)$, we add all contributions, u_P, u_T and u_R, on rays passing through the given point (t,x). The parameter ω in (3), (10) or (14) is determined from the equations of the various types of rays, (4), (7) or (10).

(6) Oscillatory Initial Data - Expansion of the Exact Solution

(6.1) The Method of Solution

In this section we shall show how the problem introduced in section (5) is solved by the method of section (2). The function $v(x, \pm\omega)$ is given by (2.11) with $r(x, \pm\omega; \lambda)$ determined by (2.6) and (5.1):

$$(1)\quad r(x, \pm\omega; \lambda) = i\lambda\left(\pm\omega + h_o(s_o'(x), x)\right) z_o(x) \exp\left[i\lambda s_o(x)\right].$$

The only new feature in our method of solution will be that u_P, u_T and u_R must now be determined by the method of two dimensional stationary phase [2]. The details of calculation will be carried out only for the function u_R. u_P and u_T can then be computed in a

similar manner. As in our previous examples, special care must be taken in finding the second derivative of the phase function associated with u_T. This problem was discussed for the Riemann function and is overcome in the same manner as was used in that case.

(6.2) The Function v_R

To find u_R we first look for that part of v which depends on $R_\pm$. This function will be called v_R and it appears as part of the solution, v, for Region I: $\omega > b_o$, $x \geq x_B$. It is given by the equation

$$(2) \quad v_R(x, \pm\omega) \sim \pm \int_{x_B}^{\infty} \left[\frac{R_\pm \exp\left[\pm i\lambda \left\{ \int_{x_B}^{\xi} + \int_{x_B}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta \right\} + i\lambda s_o(\xi)\right]}{2(\omega^2-b^2(x))^{1/4}\,(\omega^2-b^2(\xi))^{1/4}} \right.$$

$$\left. \cdot \left(\pm\omega + h_o(s_o'(\xi), \xi)\right) z_o(\xi) \exp[i\lambda s_o(\xi)] \right] d\xi ,$$

with $R_\pm$ given by (1.53).

(6.3) $u_R(t,x)$

With the above solution for v_R, we use (2.4) to define

$$(3) \quad u_R(t,x) \sim \sum_{\pm} \frac{\lambda}{2\pi} \int_{b_o}^{\infty} v_R(x, \pm\omega) \exp[\mp i\lambda\omega t]\, d\omega.$$

The phase functions of the integrands in (3) are

$$(4)\qquad \varphi_{\pm}(\omega,\ \xi) = \pm\left\{\int_{x_B}^{x} + \int_{x_B}^{\xi} \sqrt{\omega^2-b^2(\eta)}\ d\eta-\omega t\right\} + s_o(\xi)\ ,$$

with first derivatives given by

$$(5)\qquad \frac{\partial\varphi_{\pm}}{\partial\omega} = \pm\left\{\int_{x_B}^{x} + \int_{x_B}^{\xi} \frac{\omega}{\sqrt{\omega^2-b^2(\eta)}}\ d\eta - t\right\};$$

$$(6)\qquad \frac{\partial\varphi_{\pm}}{\partial\xi} = \pm\sqrt{\omega^2-b^2(\xi)} + s_o'(\xi).$$

Setting these derivatives equal to zero yields the equations

$$(7)\qquad t = \int_{x_B}^{\xi}\int_{x_B}^{x} \frac{\omega}{\sqrt{\omega^2-b^2(\eta)}}\ d\eta; \qquad s_o'(\xi) = \mp\sqrt{\omega^2-b^2(\xi)}\ .$$

Since we have assumed that $s_o'(\xi) < 0$, there can only be a stationary point for the upper sign in (7). Also, by solving for ω we obtain

$$(8)\qquad \omega = \sqrt{s_o'^2(\xi) + b^2(\xi)} = h_o(s_o'(\xi),\xi).$$

In addition to (7) and (8), it is necessary to find the determinant of the matrix of second derivatives of $\varphi_{\pm}$, $\left[\dfrac{\partial^2\varphi_{\pm}}{\partial\xi^2}\ \dfrac{\partial^2\varphi_{\pm}}{\partial\,\omega^2} - \left(\dfrac{\partial^2\varphi_{+}}{\partial\omega\partial\xi}\right)^2\right]$.

The matrix will be called $\partial^2\varphi_{\pm}$. At the stationary point defined by (7) and (8), we find, by using (5) and (6), that

$$(9)\quad \det(\partial^2\varphi_{+}) = -\frac{\omega^2}{\omega^2-b^2(\xi)} + \frac{b(\xi)b'(\xi)+s_o'(\xi)s_o''(\xi)}{\sqrt{\omega^2-b^2(\xi)}}\int_{x_B}^{\xi}+\int_{x_B}^{x}\frac{b^2(\eta)}{(\omega^2-b^2(\eta))^{3/2}}\,d\eta.$$

The first term of (9) is clearly negative for $\omega > b_o$. As in section (5) we assume that $s_o'(\xi) < 0$, $s_o''(\xi) > 0$ and throughout our discussion $b(\xi) > 0$, $b'(\xi) < 0$. Therefore $\det(\partial^2\varphi_{+}) < 0$. This implies that the eigenvalues of $(\partial^2\varphi_{+})$ are of opposite sign and hence $\mathrm{sig}(\partial^2\varphi_{+}) = 0$. Using our results, (2) —(9) and the formula for n-dimensional stationary phase in [2] we find that $u_R(t,x)$ is given parametrically by (7), (8) and

$$(10)\quad u_R \sim \frac{h_o(s_o'(\xi),\xi)R_{+}z_o(\xi)\exp\left[i\lambda\left\{s_o(\xi)+\int_{x_B}^{\xi}+\int_{x_B}^{x}\sqrt{\omega^2-b^2(\eta)}\,d\eta-\omega t\right\}\right]}{\left\{\sqrt{\omega^2-b^2(x)}\ \sqrt{\omega^2-b^2(\xi)}\ \left|\det(\partial^2\varphi_{+})\right|\right\}^{1/2}}$$

(6.4) Comparison of Results

Comparison of (7)—(10) with the solution for $u_R(t,x)$ obtained by the ray method reveals that they agree. The calculations of u_P and u_R have also been carried out and we find that they also agree with the results obtained by the ray method.

(7) The Green's Function - Ray Method

We consider now, the inhomogeneous equation

$$(1) \quad u_{xx} - u_{tt} - \lambda^2 b^2(x) u = \delta(x-x_o)\delta(t-t_o)$$

with initial data

$$(2) \quad u(t_1,x) = u_t(t_1,x) = 0; \quad t_1 < t_o.$$

At $x = x_B$ the boundary condition (1.44) is imposed.

We assume that the primary rays emanate from the point (t_o, x_o) and for each primary ray incident on the boundary a reflected ray is produced. Also, turned rays will arise as described in section (1.5). If we set

$$(3) \quad u_P = u_+ + u_-; \quad u_\pm \sim z^\pm \exp[i\lambda s^\pm] ,$$

then the functions $s^\pm$ and $z^\pm$ are determined by the methods of sections (1.2.2) and (1.3.2) except for the constants $s_o^\pm$ and $\tilde{z}^\pm$. These numbers can be determined by the indirect method.

(1.30) is the equation of the primary rays emanating from the point $(0,x_o)$. If t is replaced by $(t-t_o)$, we obtain the equation of the primary rays emanating from the point, (t_o,x_o). In fact, if this substitution is made throughout the development of the solution for the Riemann function, the corresponding components of the Green's function result.

For example, from equation (1.33), $u_P(t,x)$ for the case of constant coefficients is given parametrically by

$$(4)\quad u_P \sim \sum_{\pm} \tilde{z}^{\pm}(t-t_o)^{-1/2} \exp\left[i\lambda \left\{ s_o^{\pm} \pm \left(\sqrt{\omega^2-b^2}\,|x-x_o| - h_o(t-t_o) \right) \right\} \right];$$

$$|x-x_o| = \frac{\sqrt{\omega^2-b^2}}{h_o}(t-t_o).$$

The expansion of the exact solution for b = constant has been carried out in [2]. Section (4), equation (5) of that paper, with $a(k,\pm h_o) = 1$ and $h_o''(k) = b^2 h_o^{-3}$, yields the proper solution for comparison with (4). $s_o^{\pm}$ and $\tilde{z}^{\pm}$ are then seen to be:

$$(5)\quad s_o^{\pm} = 0, \quad \tilde{z}^{\pm} = \left(\frac{h_o}{8\pi\lambda b^2(x_o)} \right)^{1/2} e^{\mp\frac{3\pi i}{4}}.$$

Comparing (5) with (1.32) we see that they differ only in the factors $e^{\mp i\pi}$ in the equations for $\tilde{z}^{\pm}$. It therefore follows that the Green's function can be extracted from our results for the Riemann function by replacing t by $(t-t_o)$ and multiplying $\tilde{z}^{\pm}$ in the results for the latter by $e^{\mp i\pi}$.

8. The Green's Function - Expansion of the Exact Solution

In this section, we consider the asymptotic expansion of the exact solution for equation (7.1) with

$$(1)\quad \lambda^2 f(t,x;\lambda) = \delta(x-x_o)\delta(t-t_o),$$

homogeneous initial data, and the boundary condition defined by (1.44). It is now necessary to solve equation (2.6), where $r(x,\pm\omega;\lambda)$, determined from (1) and (2.7), is given by the equation

$$(2)\quad r(x,\pm\omega;\lambda) = \delta(x-x_o)e^{\pm i\lambda\omega t_o}\ .$$

By comparing (2) with equation (2.18), we see that our solutions for $v(x,\pm\omega)$ must be just multiples of the solutions for v in the expansion of the Riemann function. The multipliers are $-\exp[\pm i\lambda\omega t_o] = \exp[\mp i\pi]\exp[\pm i\lambda\omega t_o]$. If this factor is inserted in the integrals in equations (2.20) and (2.26), the functions thus obtained will be the analogous expressions for the Green's function. The constant, $\exp[\mp i\pi]$, is just a multiplier of the terms in the summations, $\sum_{\pm}$, in (2.20) and (2.26). The factor, $\exp[\pm i\lambda\omega t_o]$, has the effect of replacing t by $t-t_o$ in the integrals used to determine the Riemann function. Therefore, the same adjustment must be made in the stationary phase conditions and parametric representations of the solution. These are the same observations as were made in section (7) to calculate the expansion of the solution by the ray method. It therefore follows

that since the two methods led to the same solution for the Riemann function, the same must be true for the Green's function.

9. Diffraction in Space-Time

When finding the solution to initial-boundary value problems by the methods introduced in sections (1) and (2), all rays were confined to that portion of (t,x) space where

$$(1) \qquad t < \int_{x_B}^{x} \int_{x_B}^{x_0} \frac{b_o}{\sqrt{b_o^2 - b^2(\eta)}} \, d\eta.$$

This can be seen by examining figure (1) or the equations of the various types of rays in sections (1) - (8).

In addition, we noted in section (2) that use of the WKB method to solve the time-reduced form of (1.1) had the disadvantage of forcing us to exclude the values $\omega = \pm b_o$ from consideration. Both of these difficulties can be overcome simultaneously by a method used in [3] to treat an analogous problem for the reduced wave equation. We note that the ray defined by

$$(2) \qquad t_<(x) = \int_{x}^{x_0} \frac{b_o}{\sqrt{b_o^2 - b^2(\eta)}} \, d\eta, \quad t_< \le t(x_B);$$

$$(3) \qquad t_>(x) = \int_{x_B}^{x} + \int_{x_B}^{x_0} \frac{b_o}{\sqrt{b_o^2 - b^2(\eta)}} \, d\eta, \quad t_> > t(x_B),$$

is tangent to the space time boundary. This ray will be referred to as the critically reflected ray. Equivalently, in space, this ray is incident on the boundary at velocity $\frac{dx}{dt} = 0$. We shall show how the Riemann function is determined for $t > t_>(x)$.

We consider the initial-boundary value problem for u defined by the equations (1.1), (1.29) and (1.44). We define $v(x,\omega)$, the Fourier

transform of u, by (2.3) and its inverse transform by (2.4). Then, as shown in section (2), v is a solution of the problem defined by the equations (2.6), (2.8) and (2.9) with

(4) $$r(x,\omega;\lambda) = \delta(x-x_o).$$

As in section (2) we require that v satisfy the radiation condition

(5) $$\lim_{x\to\infty} v(x,\omega) = 0, \ \mathrm{Im}\ \omega > 0.$$

It is shown in section (2) that

(6) $$v = \frac{v_1(x_>,\omega)v_2(x_<,\omega)}{W(v_1,v_2)}\ ; \ x_> = \max\ (x,x_o),\ x_< = \min\ (x,x_o);$$

(7) $$W(v_1,v_2) = v_1v_2' - v_2v_1';$$

where v_1(or v_2) is a solution of the homogeneous form of (7) satisfying the radiation (or boundary) condition. Our goal is to obtain asymptotic expansions of v_1 and v_2(and therefore of v) which are uniformly valid through the turning point, $x_\omega{:}b(x_\omega) = \omega$. The method of obtaining these expansions is discussed in [3] and [5].

We define

(8) $$\frac{2}{3}\,[\varphi(x,\omega)]^{3/2} = \int_{x_\omega}^{x} (\omega^2 - b^2(\eta))^{\frac{1}{2}}\, d\eta = \Psi(x,\omega);\ \omega \text{ in D: } b_1^2 < \omega^2 < b_o^2$$

and assume that $\Psi(x,\omega)$ can be analytically continued in the complex ω plane. For fixed x, it is shown in Appendix III that Ψ has a branch point at $\omega = \pm\, b(x)$. We take the branch cut to run from $-\, b(x)$ to $b(x)$ and define Ψ to be positive for $b_1 > \omega > b(x)$. From [3], we define

(9) $$Y_o(x,\omega) = [\varphi'(x,\omega)]^{-\frac{1}{2}} \mathrm{Ai}[\lambda^{2/3} e^{-i\pi}\varphi(x,\omega)],$$

(10) $$Y_1(x,\omega) = [\varphi'(x,\omega)]^{-\frac{1}{2}}\mathrm{Ai}[\lambda^{2/3} e^{-\frac{i\pi}{3}} \varphi(x,\omega)].$$

Ai is the Airy function defined by

(11) $$\mathrm{Ai}(\sigma) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + \sigma t\right) dt; \ \sigma \text{ real.}$$

Y_o and Y_1 are asymptotic to two linearly independent solutions of the homogeneous equation (2.10); see [6]. It can be shown that for Re ω in D and $0 < \operatorname{Im} \omega << 1$, Y_1 satisfies the radiation condition (5). Therefore, we conclude that

(12) $$v_1(x,\omega) \sim Y_1(x,\omega).$$

Let us introduce the "boundary operator" Ω by the equation

(13) $$\Omega v(x,\omega) = v'(x_B,\omega) - i\lambda\zeta\, v(x_B,\omega).$$

It then follows that

(14) $$v_2(x,\omega) \sim Y_o\, \Omega\, Y_1 - Y_1\, \Omega\, Y_o.$$

Using the Wronskian of Airy functions given in [6] we find that

(15) $$W(v_1, v_2) = (2\pi)^{-1}(\Omega\, Y_1)\ \lambda^{2/3}\, e^{-\frac{i\pi}{6}},$$

and then (6) becomes

(16) $$v(x,\omega) \sim 2\pi\lambda^{-2/3}\, e^{\frac{i\pi}{6}}\, Y_1(x_>,\omega)\left[Y_o(x_<,\omega) - Y_1(x_<,\omega)\{\Omega Y_o\}\{\Omega Y_1\}^{-1}\right].$$

Using (9) and (10), it can be shown that

(17) $$\frac{\Omega Y_o}{\Omega Y_1} \sim \frac{\Omega\ \mathrm{Ai}[\lambda^{2/3}\, e^{-i\pi}\varphi(x,\omega)]}{\Omega\ \mathrm{Ai}[\lambda^{2/3}\, e^{-\frac{i\pi}{3}}\varphi(x,\omega)]}$$

Let us assume for the moment that the contour, Γ, in (2.4) can be "closed" in the lower half of ω plane for $t > t(x_>)$. It would then follow that

$$(18) \qquad u(t,x) = \lambda(2\pi)^{-1}\int_B v(x,\omega)\exp[i\lambda\omega t]d\omega + \sum_m \lambda i\ R_m ,$$

where B is an integral around branch cuts and on a semi-circular path in the lower half of the ω plane. The functions R_m are the residues of (18) at the zeroes of the denominator of (17). Using (9), (10), (16) and (17), we find that

$$(19) \qquad \lambda i R_m \sim \frac{2\pi\lambda^{\frac{1}{3}} e^{-\frac{i\pi}{3}} Ai[\lambda^{\frac{2}{3}} e^{-\frac{i\pi}{3}} \varphi(x,\omega_m)] Ai[\lambda^{\frac{2}{3}} e^{-\frac{i\pi}{3}} \varphi(x_o,\omega_m)]}{[\varphi'(x,\omega_m)\varphi'(x_o,\omega_m)]^{\frac{1}{2}}} \exp[-i\lambda\omega_m t]$$

$$\cdot \frac{\Omega Ai[\lambda^{\frac{2}{3}} e^{-i\pi}\varphi(x,\omega_m)]}{\frac{\partial}{\partial\omega}\left(\Omega\, Ai[\lambda^{2/3}e^{-i\pi/3}\varphi(x,\omega_m)]\right)} .$$

From the definition of Ω, we find that ω_m must satisfy the equation

$$(20) \qquad \frac{\dot{Ai}(r_m e^{-i\pi})}{Ai(r_m e^{-i\pi})} = \frac{\lambda^{\frac{1}{3}} \zeta e^{5\pi i/6}}{\varphi'(x_B,\omega_m)} ;$$

$$(21) \qquad r_m = \lambda^{2/3} e^{2\pi i/3} \varphi(x_B,\omega_m).$$

It can be verified that for each ω_m such that $\mathrm{Re}\omega_m > 0$, its image, obtained by reflection through the imaginary axis, $\mathrm{Re}\omega_m \to -\mathrm{Re}\omega_m$, is also a root. We will therefore consider only those roots where $\mathrm{Re}\omega_m > 0$ and indicate later how the results are to be adjusted to include residues R_m for which $\mathrm{Re}\omega_m < 0$.

Let us define $x_m = x_{\omega_m}$. From (8) and (21) we see that for r_m

to be finite as $\lambda \to \infty$, $x_m \sim x_B$. In this case $\varphi(x_B, \omega_m)$ and $\psi(x_B, \omega_m)$ can be approximated by integrating by parts in (8). By this process it can be shown that

$$\omega_m \sim b_o + r_m(2b_o)^{-1/3}(b_o'\lambda^{-1})^{2/3} e^{-i\pi/3}; \tag{22}$$

$$b_o' = -b'(x_B) > 0. \tag{23}$$

It can also be shown that asymptotically (20) becomes

$$\frac{\dot{A}i(r_m e^{-i\pi})}{Ai(r_m e^{-i\pi})} = \frac{\zeta e^{5\pi i/6}}{(2b_o b_o'\lambda^{-1})^{1/3}} \,. \tag{24}$$

In addition, we may simplify the denominator of (19) by noting that

$$\frac{\partial}{\partial \omega} \Omega Ai(r_m e^{-i\pi}) \sim 2^{1/3}(b_o b_o'\lambda^{-1})^{-2/3}\, b_o \Omega \dot{A}i(r_m e^{-i\pi}). \tag{25}$$

The Airy functions appearing in (19) can be expanded asymptotically by using the results in [6]. We then find that

$$\lambda i R_m \sim \left(\frac{2b_o b_o'}{\lambda}\right)^{2/3} \frac{e^{\pi i/6}\ \Omega Ai(r_m e^{-5\pi i/3})}{2b_o \Omega Ai(r_m e^{-i\pi})} \cdot \frac{\exp\left[i\lambda\left\{\left(\int_{x_m}^{x} + \int_{x_m}^{x_o}\right)(\omega_m^2 - b^2(\eta))^{\frac{1}{2}}\, d\eta - \omega_m t\right\}\right]}{[\varphi^{\frac{1}{2}}(x,\omega_m)\varphi'(x,\omega_m)\varphi^{\frac{1}{2}}(x_o,\omega_m)\varphi'(x_o,\omega_m)]^{\frac{1}{2}}} \,. \tag{26}$$

The denominator in (26) can be further simplified by differentiating (8) and evaluating the result at x and x_o. In addition, the exponent in (26) can be expanded asymptotically by the same procedure as appears in [3].

$$(27) \qquad i\lambda\left\{\int_{x_m}^{x} + \int_{x_m}^{x_o} (\omega_m^2 - b^2(\eta))^{\frac{1}{2}} d\eta - \omega_m t\right\} \sim i\lambda\left\{\int_{x_B}^{x} + \int_{x_B}^{x_o} (b_o^2 - b^2(\eta))^{\frac{1}{2}} d\eta - b_o t\right\}$$

$$+ r_m (2b_o)^{-1/3} (b_o')^{2/3} \lambda^{1/3} e^{5\pi i/6} (T_B - t_B) - \frac{4i}{3} r_m^{3/2}$$

where

$$(28) \qquad t_B = \int_{x_B}^{x_o} \frac{b_o}{\sqrt{b_o^2 - b^2(\eta)}} d\eta \; ; \; t - T_B = \int_{x_B}^{x} \frac{b_o}{\sqrt{b_o^2 - b^2(\eta)}} d\eta .$$

t_B is the quantity previously defined as $t_<(x_B)$, the arrival time of the critically reflected ray at the boundary. We interpret T_B as a time of emission of a new type of ray which we will call the <u>time-diffracted ray</u>. The requirement that $t > t_>(x)$ implies that $T_B > t_B$ and we see that the time-diffracted ray is just the critically reflected ray displaced in space-time by the amount $T_B - t_B$. (See figure (1)).

We may now rewrite (26) as

$$(29) \qquad \lambda i R_m \sim \left(\frac{2b_o b_o'}{\lambda}\right)^{2/3} \frac{e^{\pi i/6} \Omega Ai(r_m e^{-5\pi i/3}) \exp\left[i\lambda\left\{\int_{x_B}^{x} + \int_{x_B}^{x_o} \sqrt{b_o^2 - b^2(\eta)}\, d\eta - b_o t\right\}\right]}{2b_o \Omega \dot{Ai}(r_m e^{-i\pi}) [b_o^2 - b^2(x)]^{\frac{1}{2}} [b_o^2 - b^2(x_o)]^{\frac{1}{2}}}$$

$$\cdot \exp\left\{r_m (\lambda b_o'^2 [2b_o]^{-1})^{1/3} e^{5\pi i/6} (T_B - t_B) - \frac{4i}{3} r_m^{3/2}\right\} .$$

For the special case where $\zeta \to \infty$ in (24), $r_m > 0$ [6]. In this case from (29) R_m decays exponentially with $\lambda^{1/3} r_m (T_B - t_B)$. A further simplification is possible if we assume that $r_m = O(\lambda^\alpha)$; $0 < \alpha < 1$. In this case we can use the Wronskian of the Airy functions [6] and the boundary condition (13) to simplify the quotient of boundary operators in (29). The result is

$$(30) \qquad \frac{\Omega Ai(r_m e^{-5\pi i/3})}{\Omega A\dot{i}(r_m e^{-i\pi})} \sim \frac{\lambda^{2/3} e^{-i\pi/6}}{2\pi (2b_o b_o')^{1/3}} \left[Ai(r_m e^{-i\pi}) \Omega A\dot{i}(r_m e^{-i\pi}) \right]^{-1}$$

By using (13, (24) and the asymptotic form of $\varphi'(x_B, \omega_m)$, we simplify this expression still further and obtain

$$(31) \qquad \frac{\Omega Ai(r_m e^{-5\pi i/3})}{\Omega A\dot{i}(r_m e^{-i\pi})} \sim \left\{ 2\pi e^{\pi i/2} \left[(2b_o b_o')^{2/3} r_m e^{\pi i/3} + \lambda^{2/3} \zeta^2 \right] Ai^2(r_m e^{-i\pi}) \right\}^{-1}.$$

Substitution of (31) into (29) yields, for large values of r_m, the expression

$$(32) \qquad \lambda i R_m \sim \left(\frac{2b_o b_o'}{\lambda} \right)^{2/3} \frac{e^{-i\pi/3} \exp\left[i\lambda \left\{ \left(\int_{x_B}^{x} + \int_{x_B}^{x_o} \right) \sqrt{b_o^2 - b^2(\eta)}\, d\eta - b_o t \right\} \right]}{4\pi b_o \left[b_o^2 - b^2(x) \right]^{1/4} \left[b_o^2 - b^2(x_o) \right]^{1/4}}$$

$$\cdot \frac{\exp\left\{ r_m (\lambda b_o'^2 [2b_o]^{-1})^{1/3} e^{5\pi i/6} (T_B - t_B) - \frac{4i}{3} r_m^{2/3} \right\}}{\left[(2b_o b_o')^{2/3} r_m e^{\pi i/3} + \lambda^{2/3} \zeta^2 \right] Ai^2(r_m e^{-i\pi})}.$$

We show in appendix III that the branch cut integrals and the integrals along the semi-circular path in the lower half plane do not contribute to the result and therefore from (22)

(33) $$u(t,x) \sim \sum_{m=1}^{\infty} \lambda i R_m$$

where R_m is given by (29) for $r_m = O(1)$ and by (32) for $r_m = O(\lambda^{\alpha})$; $0 < \alpha < 1$.

To find R_m for $\mathrm{Re}\,\omega_m < 0$, we introduce the following rotation

(34) $$\mathcal{T}\left[f(\omega_1 + i\omega_2)\right] = f(-\omega_1 + i\omega_2).$$

It can then be shown that

(35) $$\mathcal{T}(\lambda i R_m) = \overline{\lambda i R_m}\, e^{-\frac{2\pi i}{3}} \quad ; \quad r_m \to \infty.$$

Application of (35) to (32) allows us to find the residues for $\mathrm{Re}\,\omega_m < 0$. For $r_m = O(1)$, we replace

$$\Omega \mathrm{Ai}\left(r_m e^{-\frac{5\pi i}{3}}\right) \quad \text{by} \quad \Omega \mathrm{Ai}\left(r_m e^{-\frac{4\pi i}{3}}\right) \quad \text{in (29)}$$

and call the new expression $\tilde{R}_m$. It can be shown that

(36) $$\mathcal{T}(\lambda i R_m) = \overline{\lambda i \tilde{R}_m}\, e^{-\frac{i\pi}{3}} \quad ; \quad r_m = O(1).$$

Appendix I

The purpose of this appendix is to find two linearly independent solutions to the homogeneous ordinary differential equation, (2.10). One solution, $v_2(x,\pm\omega)$, should satisfy the boundary condition, (2.8), and the other, $v_1(x,\pm\omega)$, should satisfy the radiation condition, $\lim_{x\to\infty} v(x,\omega) = 0$, $\operatorname{Im}\omega > 0$.

<u>Part 1</u>: $v_1(x,\pm\omega)$, $v_2(x,\pm\omega)$ for $\omega > b_o$.

By formal substitution into (10) of a function in the form $a(x,\omega)\exp i\lambda[s(x,\omega)]$, one finds that two linearly independent solutions have asymptotic expansions given by

$$(1) \qquad v^{(1)}(x,\omega) = (\omega^2-b^2(x))^{-\frac{1}{4}} \exp[i\lambda \int_{x_B}^{x} (\omega^2-b^2(\eta))^{\frac{1}{2}} d\eta],$$

$$(2) \qquad v^{(2)}(x,\omega) = (\omega^2-b^2(x))^{-\frac{1}{4}} \exp[-i\lambda \int_{x_B}^{x} (\omega^2-b^2(\eta))^{\frac{1}{2}} d\eta].$$

To find a solution satisfying the radiation condition, (2.9), we must extend $v^{(1)}$ and $v^{(2)}$ into the upper half ω plane and examine them for $\operatorname{Im}\omega > 0$ and $x \to \infty$.

For complex ω, the asymptotic behavior of $v^{(1)}$ and $v^{(2)}$ depend on the nature of the exponents. We define

$$(3) \qquad e_{\pm}(\omega) = \pm\int_{x_B}^{x} (\omega^2 - b^2(\eta))^{\frac{1}{2}} d\eta.$$

The function, $(\omega^2 - b^2(\eta))^{\frac{1}{2}}$, is defined as an analytic function of ω in the "cut plane", with branch cut from $-b(\eta)$ to $b(\eta)$. Also

$$(4) \qquad (\omega^2 - b^2(\eta))^{\frac{1}{2}} = \begin{cases} \sqrt{\omega^2 - b^2(\eta)}\ , & \omega > b(\eta) \\ \sqrt{\omega^2 - b^2(\eta)}\ , & -\omega < b(\eta) \end{cases} .$$

For $\omega = \omega_o + i\,\omega_1$ and $\omega_1 > 0$ but small, $e_\pm(\omega)$ can be approximated by

$$(5) \qquad e_\pm(\omega) \approx e_\pm(\omega_o) + e'_\pm(\omega_o)(\omega - \omega_o).$$

With this approximation, we find that

$$(6) \qquad \operatorname{Im} e_+(\omega) > 0,\ \operatorname{Im} e_-(\omega) < 0;\ \operatorname{Im}\omega > 0.$$

We conclude that $v^{(1)}(x,\omega)$ decays exponentially as $x \to \infty$ and $v^{(2)}(x,\omega)$ grows exponentially as $x \to \infty$. By using the definition of $(\omega^2 - b^2)^{\frac{1}{2}}$ in terms of $\sqrt{\omega^2 - b^2}$ we rewrite (1) and (2) as

$$(7) \qquad v^{(1)}(x,\pm\omega) \sim (\omega^2 - b^2(x))^{-\frac{1}{4}} \exp\left[\pm i\lambda \int_{x_B}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta\right],$$

$$(8) \qquad v^{(2)}(x,\pm\omega) \sim (\omega^2 - b^2(x))^{-\frac{1}{4}} \exp\left[\mp i\lambda \int_{x_B}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta\right];\ \omega > b_o.$$

The conclusion of the analysis above is that the function,

$$(9) \qquad v_1(x,\pm\omega) = v^{(1)}(x,\pm\omega),$$

satisfies the radiation condition . In (7) and (8), one should also define the factor $(\omega^2 - b^2(x))^{-\frac{1}{4}}$ in terms of the positive fourth root, $\sqrt[4]{\omega^2 - b^2}$. Any phase factors introduced by this definition will also appear in the Wronskian and would not affect $v(x,\omega)$ as defined by (2.11). We therefore interpret the amplitudes as positive fourth roots.

The function, $v_2(x,\pm\omega)$, satisfying the boundary condition, is given by the following linear combination of $v^{(1)}$ and $v^{(2)}$:

$$(10) \qquad v_2(x,\pm\omega) = \pm\left[v^{(1)}(x,\pm\omega)\exp[\pm i\gamma] - v^{(2)}(x,\pm\omega)\exp[\mp i\gamma]\right],\ \omega > b_o.$$

γ is determined by substituting (10) into the boundary condition, (2.8). For this purpose, the following form of (10) is simpler to use:

(11) $$v_2(x,\pm\omega) \sim 2i(\omega^2 - b^2(x))^{-\frac{1}{4}} \sin\left[\lambda \int_{x_B}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta + \gamma\right].$$

γ is then determined by the equation,

(12) $$\cot\gamma = \frac{i\zeta}{\sqrt{\omega^2 - b_o^2}} \quad ; \; b_o = b(x_B).$$

The Wronskian, $v_1 v_2' - v_2 v_1'$, can be obtained by using (9) and (10). The result is

(13) $$W(v_1, v_2) \sim 2i\lambda \exp[\mp i\gamma].$$

<u>Part 2</u>: $v_1(x,\pm\omega)$, $v_2(x,\pm\omega)$ for $b_o > \omega > b_1$

For each ω in this range, there exists a turning point, that is an $x = x_\omega$, such that $\omega^2 - b^2(x) > 0$ for $x > x_\omega$ and $\omega^2 - b^2(x) < 0$ for $x < x_\omega$. $v_1(x,\pm\omega)$ is determined by its behavior for x large. In particular, for each fixed ω, we can consider the range $x > x_\omega$ to determine this function. The analysis of part (1) is simply repeated with x_B replaced by x_ω and the result for $x > x_\omega$ (analogous to (7) and (9)) is

(14) $$v_1(x,\pm\omega) \sim (\omega^2 - b^2(x))^{-\frac{1}{4}} \exp\left[\pm i\lambda \int_{x_\omega}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta\right]; \; x > x_\omega.$$

$v_2(x,\pm\omega)$ is determined by its behavior at $x = x_B < x_\omega$, where $\omega^2 - b^2(x) < 0$. Analogous to equation (11) we may write

(15) $$v_2(x,\ \omega) \sim 2(b^2(x) - \omega^2)^{-\frac{1}{4}} \sinh\left[\lambda \int_{x_B}^{x} \sqrt{b^2(\eta) - \omega^2} + \gamma\right].$$

By using the boundary condition, (2.8), we find that

$$(16) \qquad \coth \gamma = \frac{i\zeta}{\sqrt{b^2 - \omega^2}}$$

In order to determine $v_2(x, \pm\omega)$ for $x > x_\omega$ we must use the WKB connection formulas [6,8]. For a homogeneous second order ordinary differential equation, these formulas are asymptotic expansions of a specific pair of linearly independent solutions for $x < x_\omega$ and $x > x_\omega$. The connection formulas for (2.10) are given below.

$$(17) \qquad v^{(3)}(x) \sim (\omega^2 - b^2(x))^{-\frac{1}{4}} \sin\left[\lambda \int_{x_\omega}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta + \frac{\pi}{4}\right], \quad x > x_\omega;$$

$$(18) \qquad v^{(3)}(x) \sim \tfrac{1}{2}(b^2(x) - \omega^2)^{-\frac{1}{4}} \exp\left[-\lambda \int_{x}^{x_\omega} \sqrt{b^2(x) - \omega^2}\, d\eta\right], \quad x < x_\omega.$$

$$(19) \qquad v^{(4)}(x) \sim (\omega^2 - b^2(x))^{-\frac{1}{4}} \cos\left[\lambda \int_{x_\omega}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta + \frac{\pi}{4}\right], \quad x > x_\omega;$$

$$(20) \qquad v^{(4)}(x) \sim (b^2(x) - \omega^2)^{-\frac{1}{4}} \exp\left[\lambda \int_{x}^{x_\omega} \sqrt{b^2(x) - \omega^2}\right], \quad x < x_\omega.$$

To express $v_2(x, \pm\omega)$ in terms of $v^{(3)}$ and $v^{(4)}$, we note first that

$$(21) \qquad \int_{x_B}^{x} \sqrt{b^2(\eta) - \omega^2}\, d\eta = \int_{x_B}^{x_\omega} \sqrt{b^2(\eta) - \omega^2}\, d\eta - \int_{x}^{x_\omega} \sqrt{b^2(\eta) - \omega^2}\, d\eta.$$

Using the relations between sinh and exponential functions, (15) may be rewritten as

$$(22) \qquad v_2(x, \pm\omega) \sim 2\beta\, v^{(3)}(x) - \beta^{-1}\, v^{(4)}(x)$$

where

$$\beta = \exp\left[\lambda \int_{x_B}^{x_\omega} \sqrt{b^2(\eta) - \omega^2}\, d\eta + \gamma\right] \tag{23}$$

For $x > x_\omega$, then, the solution v_2 can be obtained from the connection formulas:

$$v_2(x, \pm\omega) \sim (\omega^2 - b^2)^{-\frac{1}{4}} \; 2\beta \sin\left[\lambda \int_{x_\omega}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta + \frac{\pi}{4}\right] + \tag{24}$$

$$\beta^{-1} \cos\left[\lambda \int_{x_\omega}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta + \frac{\pi}{4}\right].$$

To simplify the determination of the Wronskian we note that from (14)

$$v_1(x, \pm\omega) \sim e^{\frac{\mp i\pi}{4}} \left[\pm i\, v^{(3)}(x) + v^{(4)}(x)\right] \tag{25}$$

Then, from (22) and (25),

$$W(v_1, v_2) \sim e^{\frac{\mp i\pi}{4}}\, W\left[\pm i v^{(3)}(x) + v^{(4)}(x),\; 2\beta v^{(3)}(x) - \beta^{-1} v^{(4)}(x)\right]; \tag{26}$$

$$W(v_1, v_2) \sim (2\beta \pm i\beta^{-1})\, e^{\frac{\mp i\pi}{4}}\, W(v^{(4)}, v^{(3)}) \sim \lambda(2\beta \pm i\beta^{-1})\, e^{\frac{\mp i\pi}{4}},$$

The final result is obtained from (17) and (19) or from (18) and (20). Noting that β is exponentially large in λ, the quotient, $\frac{v_2}{W}$, is formed and the exponentially small part neglected. It follows that for $x > x_\omega$

$$\frac{v_2(x, \pm\omega)}{W(v_1, v_2)} \sim \frac{\left[\omega^2 - b^2(x)\right]^{-\frac{1}{4}} e^{\frac{\pm i\pi}{4}}}{2i\lambda} \left\{\exp\left[i\lambda \int_{x_\omega}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta + \frac{i\pi}{4}\right] - \right. \tag{27}$$

$$\left. \exp\left[-i\lambda \int_{x_\omega}^{x} \sqrt{\omega^2 - b^2(\eta)}\, d\eta - \frac{i\pi}{4}\right]\right\}.$$

Appendix II

In this section, we prove that the integral given by (1), below, is asymptotically equal to the integral given by (3). This result allows us to simplify equations (4.4), (4.5) and (4.11).

We consider the integral,

(1) $$I(\lambda) = \lambda \int_{-\infty}^{\infty} f(z) g(\lambda z) \exp[i\lambda \int_{0}^{z} \varphi(\eta) d\eta] \, dz.$$

If $g(z)$ is a continuous function with compact support (call the support $\mathcal{D}$; $|z| \leq z_0$ for $z \in \mathcal{D}$) and φ and f have continuous derivatives in $\mathcal{D}$, then

(2) $$I(\lambda) = K + O\left(\frac{1}{\lambda}\right) ,$$

where

(3) $$K = f(0) \int_{-\infty}^{\infty} g(z) \exp[i\varphi(0) z] \, dz.$$

Proof:

(4) $$I(\lambda) - K = \lambda \int_{-\infty}^{\infty} g(\lambda z) \left\{ f(z) \exp[i\lambda \int_{0}^{z} \varphi(\eta) d\eta] - f(0) \exp[i\lambda\varphi(0) z] \right\} dz.$$

We rewrite (4) as

(5) $$I(\lambda) - K = I_2(\lambda) + I_3(\lambda),$$

where

(6) $$I_2(\lambda) = \lambda \int_{-\infty}^{\infty} g(\lambda z) f(z) \left\{ \exp[i\lambda \int_{0}^{z} \varphi(\eta) d\eta] - \exp[i\lambda\varphi(0) z] \right\} dz;$$

$$(7) \qquad I_3(\lambda) = \lambda \int_{-\infty}^{\infty} g(\lambda z) \exp[i\lambda\varphi(o)z]\left\{f(z) - f(o)\right\} dz.$$

Let

$$(8) \qquad F_1 = \max_{z\in\mathfrak{D}} |f(z)|, \; F_2 = \max_{z\in\mathfrak{D}} |f'(z)|, \; G = \max_{z\in\mathfrak{D}} |g(z)|, \; \Phi = \max_{z\in\mathfrak{D}} |\varphi'(z)|.$$

From (6) and (8)

$$(9) \qquad |I_2(\lambda)| \leq 2\lambda F_1 G \int_{|\lambda z| \leq z_o} \left| \sin \frac{\lambda}{2} \int_o^z (\varphi(\eta) - \varphi(o) d\eta \right| dz$$

We introduce the variables ξ and ζ, defined by

$$(10) \qquad \xi = \lambda\eta \quad \zeta = \lambda z,$$

and rewrite (9) as

$$(11) \qquad |I_2(\lambda)| \leq 2 \; F_1 G \int_{|\zeta| \leq z_o} \left| \sin \tfrac{1}{2} \int_o^{\zeta} \left(\varphi\left(\frac{\zeta}{\lambda}\right) - \varphi(o)\right) d\xi \right| d\zeta.$$

Since $|\sin\theta| \leq |\theta|$,

$$(12) \qquad |I_2(\lambda)| \leq F_1 G \int_{|\zeta| \leq z_o} \left| \int_o^{\zeta} \left(\varphi\left(\frac{\xi}{\lambda}\right) - \varphi(o)\right) d\xi \right| d\zeta;$$

$$(13) \qquad I_2(\lambda) \leq F_1 G \int_{|\zeta| \leq z_o} \int_o^{\zeta} \Phi \frac{|\xi|}{\lambda} \; d\xi \; d\zeta = \frac{F_1 G \,\Phi\, z_o^3}{3\lambda}.$$

Letting $\zeta = \lambda z$ in (7), we obtain

$$(14) \qquad I_3(\lambda) = \int_{\zeta\in\mathfrak{D}} g(\zeta) \exp[i\varphi(o)z]\left\{f\left(\frac{\zeta}{\lambda}\right) - f(o)\right\} d\zeta;$$

$$(15) \qquad |I_3(\lambda)| \leq G \int_{|\zeta| \leq z_o} \left| f\left(\frac{\zeta}{\lambda}\right) - f(o) \right| d\zeta \leq G \int_{|\zeta| \leq z_o} F_2 \frac{|\zeta|}{\lambda} \, d\zeta.$$

Then

$$(16) \qquad |I_3(\lambda)| \leq \frac{G\ F_2}{\lambda} z_o^2$$

From (4), (13) and (16),

$$(17) \qquad |I(\lambda) - K| \leq \frac{C}{\lambda} \ ,$$

so that

$$(18) \qquad I(\lambda) = K + O\left(\frac{1}{\lambda}\right) .$$

Appendix III

In this appendix we shall show that the branch cut integral in (9.18) is asymptotically zero. Also, the integral on a semicircle in the lower half of the ω plane tends to zero with increasing radius.

In order to determine the branch points and branch cuts of the integrand of (9.18) we must examine the functions $\varphi(x,\omega)$ and $\psi(x,\omega)$ defined by (9.8). For $b(x) < \omega < b_o$, $\psi(x,\omega) > 0$ and $\arg \psi(x,\omega) = 0$. When ω is near $b(x)$, by integrating by parts in (9.12) we find that

$$(1) \quad \psi(x,\omega) \approx \frac{\left(\omega^2 - b^2(x)\right)^{3/2}}{-b(x)b'(x)} \quad ; \quad \omega \approx b(x).$$

Hence ψ has a branch point of order 3/2 at $\pm b(x)$. At $\omega = \pm b_o$ and $\omega = \pm b_1$ the functions φ and ψ can be analytically continued by power series and therefore have no singularity. The functions $\varphi(x, \omega)$, $\varphi(x_B,\omega)$, $\varphi(x_o,\omega)$ and their derivatives with respect to x appear in (9.16). We therefore have branch points at $\pm b(x)$, $\pm b(x_o)$ and $\pm b_o = \pm b(x_B)$ and take the branch cuts to extend from $-b(x)$ to $b(x)$, $-b(x_o)$ to $b(x_o)$, and $-b_o$ to b_o. The integral around branch cuts is then an integral enclosing the real ω axis from $-b_o$ to b_o.

From (1) we see that $\psi(\tilde{x},\omega)$, $\tilde{x} = x$, x_o, or x_B, has a branch point of order 3/2 at $\tilde{x}$. Let us consider $\tilde{x} = x$. For $\omega > b(x)$, $\psi(x,\omega) > 0$ and $\varphi(x,\omega) > 0$; $\arg \psi = \arg \varphi = 0$. From (1), for $b_1 < \omega < b(x)$ and above the cut, $\arg \psi = \frac{3\pi}{2}$ and, from (9.8), $\arg \varphi = \pi$. Below the cut, $\arg \psi = -\frac{3\pi}{2}$ and $\arg \varphi = -\pi$. Hence for $-b_1 < \omega < b(x)$ $\arg \varphi$ differs by 2π above and below the cut. Since the Airy function is entire, $Ai\left(\lambda^{2/3} e^{-i\pi} \varphi(x,\omega)\right)$ and $Ai\left(\lambda^{2/3} e^{-\frac{i\pi}{3}} \varphi(x,\omega)\right)$ are the same above and below the cut. Since $\arg \varphi$

does not change at b_1, this must be true also for $-b_1 < \omega < b_1$ and for $-b(x) < \omega < b(x)$. By a similar analysis, for $-b_o < \omega < -b(x)$, arg φ differs by 4π above and below the cut and hence again the Airy functions are the same. A similar analysis can be carried out for $\tilde{x} = x_o$ or x_B. To analyze the derivative with respect to x, we note first that from (1) and (9. 8)

$$(2) \qquad \varphi(x,\omega) \approx (\tfrac{3}{2})^{2/3} \frac{\omega^2 - b^2(x)}{\left(-b(x)b'(x)\right)^{2/3}}; \quad \omega \approx b(x).$$

By differentiating (9. 8) with respect to x, it follows that

$$(3) \quad \varphi'(x,\omega)\left[\varphi(x,\omega)\right]^{1/2} = \left[\omega^2 - b^2(x)\right]^{1/2}.$$

From (2) and (3) it follows that arg $\varphi'(x,\omega) \equiv 0$. This analysis also holds for $\tilde{x} = x_o$, x_B. Consequently, since the paths of integration above and below the cut are in opposite directions it is clear that the resulting integral is zero.

To analyze the integral on the semicircle in the lower half plane, we note first that for $b_o^2 < \omega^2$(real) there are no turning points. On the real line then, the solution is given by $v(x, +\omega)$ in (2.19) with R_+ replaced by the quotient:

$$(4) \quad \tilde{R} = \Omega \, Ai\left[\lambda^{\frac{2}{3}} e^{-i\pi} \varphi(x_B,\omega)\right] \left\{ \Omega Ai\left[\lambda^{\frac{2}{3}} e^{-\frac{i\pi}{3}} \varphi(x_B,\omega)\right] \right\}^{-1}.$$

On a semicircle of sufficiently large radius in the lower half ω-plane it can be shown that $\tilde{R}$ is bounded. In fact, to show that the integral on this semicircle is zero we need only show that the exponents

$$(5)\quad e^{(1)}_{\pm} = \pm \int_{x_B}^{x} + \int_{x_B}^{x_o} \left(\omega^2 - b^2(\eta)\right)^{1/2} d\eta - \omega t$$

and

$$(6)\quad e^{(2)}_{\pm} = \pm \int_{x_<}^{x_>} \left(\omega^2 - b^2(\eta)\right)^{1/2} d\eta - \omega t$$

have a positive imaginary part for Im $\omega < 0$. In this case, the integrand of (9.18) will exponentially decay in the lower half plane as $|\omega| \to \infty$. As a sample, let us consider

$$(7)\quad e^{(1)}_{+} = \int_{x_B}^{x} + \int_{x_B}^{x_o} \left(\omega^2 - b^2(\eta)\right)^{1/2} d\eta - \omega t.$$

From (9.28) for the region of interest

$$(8)\quad e^{(1)}_{+} = \int_{x_B}^{x} + \int_{x_B}^{x_o} \left[\left(\omega^2 - b^2(\eta)\right)^{1/2} - \frac{\omega b_o}{\left(b_o^2 - b^2(\eta)\right)^{1/2}}\right] d\eta - \omega(T_B - t_B).$$

We note that $T_B - t_B > 0$, so that $\mathrm{Im}[-\omega(T_B - t_B)] > 0$. The integral, I, appearing in (8) can be rewritten as

$$(9)\quad I = \omega \int_{x_B}^{x} + \int_{x_B}^{x_o} \left[\left(1 - \frac{b^2(\eta)}{\omega^2}\right)^{1/2} \left(b_o^2 - b^2(\eta)\right)^{1/2} - b_o \right] \left(b_o^2 - b^2(\eta)\right)^{-1/2} d\eta.$$

As $|\omega| \to \infty$

$$(10)\quad \left(1 - \frac{b^2(\eta)}{\omega^2}\right)^{1/2} \approx 1 - \frac{1}{2}\frac{b^2(\eta)}{\omega^2}$$

so that

$$(11)\quad I \approxeq \omega \int_{x_B}^{x} + \int_{x_B}^{x_o} \left[\left(b_o^2 - b^2(\eta)\right)^{1/2} - b_o - \frac{1}{2}\frac{b^2(\eta)}{\omega^2} \left(b_o^2 - b^2(\eta)\right)^{\frac{1}{2}} \right] \left(b_o^2 - b^2(\eta)\right)^{-\frac{1}{2}} d\eta.$$

But

$$(12)\quad \left(b_o^2 - b^2(\eta)\right)^{\frac{1}{2}} - b_o < 0$$

so that for $\mathrm{Im}\,\omega < 0$

(13) $\text{Im } I > 0$

for $|\omega|$ sufficiently large. Hence Im $e_+^{(1)} > 0$ and Re $ie_+^{(1)} < 0$ in the lower half plane. A similar analysis can be carried out for all exponents in (5) and (6). We then conclude that asymptotically the integrand of (9.18) decays exponentially as $|\omega| \to \infty$ in the lower half-plane. On the real axis the integrand also approaches zero as $|\omega| \to \infty$. With this information it can be shown that the integral on a semi-circle in the lower half plane approaches zero as the radius of the circle approaches ∞.

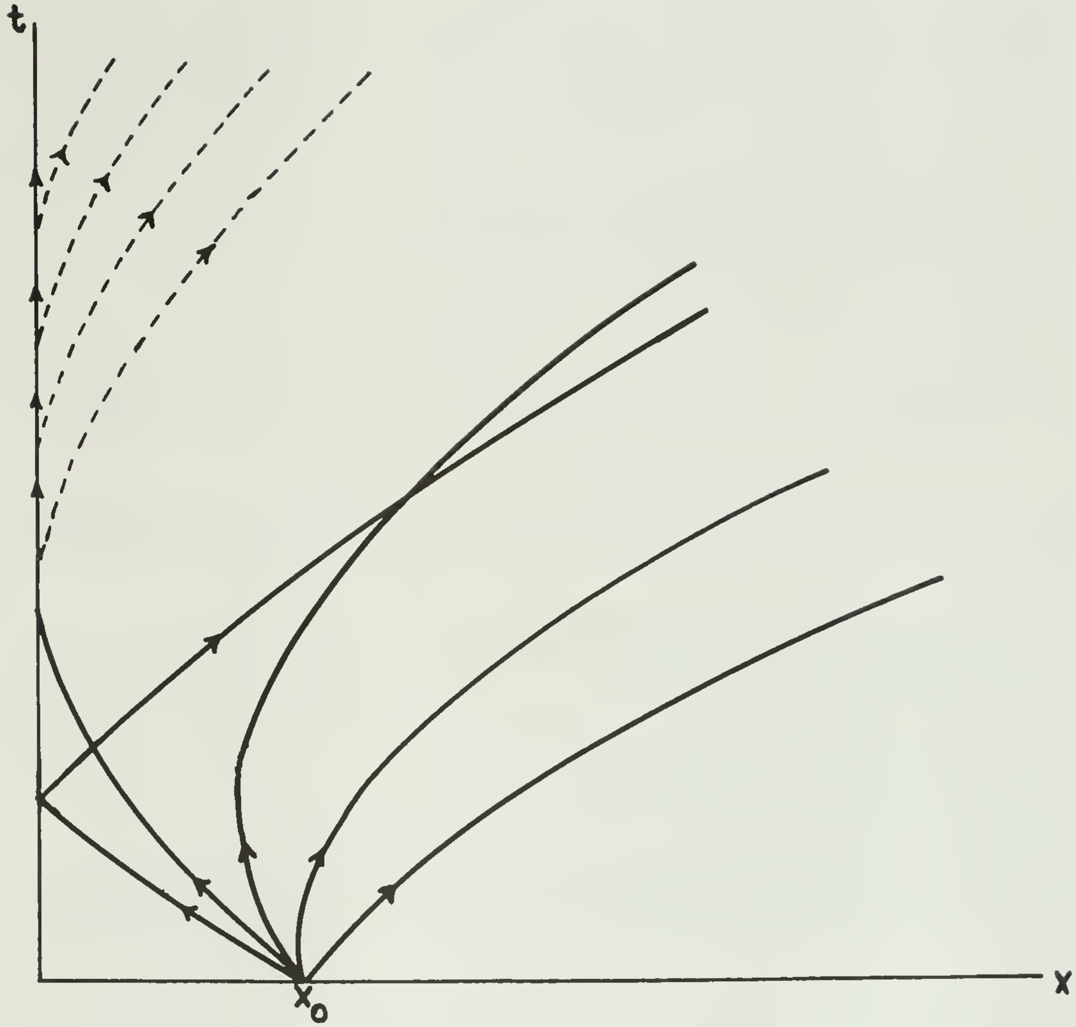

Fig. 1

The ray picture for the case of rapidly varying initial data.

_______________ Primary, turned and reflected rays.

--------------- Space-time diffracted rays.

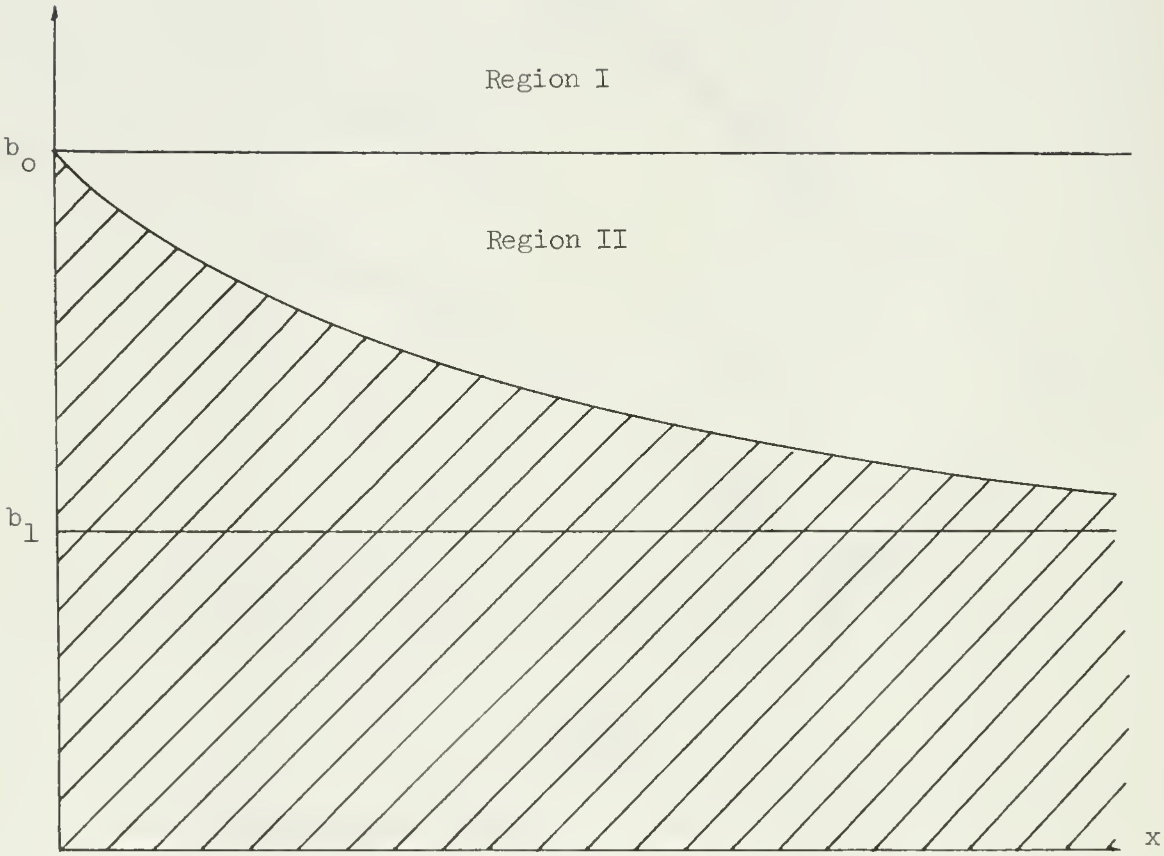

Fig. 2 b(x)

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13. ABSTRACT

In this paper we consider initial-boundary value problems for an energy conserving dispersive hyperbolic equation, the Klein-Gordon equation. This equation contains the main feature of dispersion: The speed of propagation depends on the frequency.

The primary purpose of this paper is to compare the asymptotic expansion of solutions obtained by a technique which we call the ray method with the asymptotic expansion of the exact solution. In every case considered, the solutions agree.

Solutions are obtained for a series of initial-boundary value problems in one space dimension with variable coefficients. As a by-product of our work we find a new feature which we call space-time diffraction. This phenomenon has the following physical interpretation: A portion of the energy of a wave reaches a boundary surface and then gradually leaks off, leaving a diminishing residue on the boundary for all time.

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